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UNDERGROUND EXPLOSION THEORY

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UNDERGROUND EXPLOSION THEORY

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PROJECT 1.9

THEORETICAL STUDIES OF UNDERGROUND SHOCK WAVES

by

C.B. Morrey, Jr., Edmund Pinney
R.G. Stoneham, P.L. Chambre
R.M. Lakness, and Emanuel Parzen

April 1952

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ABSTRACT

This report contains all the completed results obtained in connection with Project 1.9 of Operation JANGLE by those working on Project NR 340-040, Contract Nonr 222-04, entitled "Shock Waves in Solids". The report consists of several chapters of which the first contains a general discussion of the work done and each of the following chapters is a complete study in itself. The authors of the chapters are as follows:

Chapter 2: Professor Morrey, Mr. Parzen, Dr. Lakness;
Chapter 3: Professor Pinney;
Chapter 4: Dr. Stoneham;
Chapter 5: Professor Morrey;
Chapter 6: Dr. Chambré.

Besides containing a general discussion of all the results, Chapter 1 contains a discussion (see Paragraph 1.3.2) in support of our belief that the ground behaves like an elastic solid at distances from the explosion corresponding to values of the scaled distance (for definition, see §1.1) which are greater than 4. The necessary mathematical study has not yet been completed.

Chapter 2 derives the most general possible relation between the stress tensor T and the strain tensor E which can hold in an isotropic medium. It is assumed merely that the components of T are single-valued functions of the components of E only. In matrix notation (see §2.2 for notations, etc.), the result is as follows: There exist three scalar functions $\varphi_0(L,M,N)$, $\varphi_1(L,M,N)$, and $\varphi_2(L,M,N)$ of the strain tensor such that

$$T = \varphi_0(L,M,N)\ I + \varphi_1(L,M,N)\ E + \varphi_2(L,M,N)\ E^2.$$

In Chapter 3, a theory is developed of a hypothetical material whose mechanical behavior may approximate that of soil. The material differs from an elastic material in that a Coulomb friction mechanism is postulated which permits plastic yield when shearing stress becomes too high with respect to compressive stress. The material cannot support tensile stress when dry. Rough corrections to take into account the presence of moisture are given. The theory is applied to one-dimensional problems of wave

propagation. An interesting result is that at a finite time after the energy source cuts off, all motion ceases.

Chapter 4 presents methods, which use the complex inversion integral of the Laplace transform, whereby one can obtain the exact formal solution for the displacements in an elastic half-space due to any arbitrary radial pressure-time distribution on the surface of a small finite spherical cavity within the half-space.

Chapter 5 presents a derivation using methods of statistical mechanics of the equations of mass-motion of a medium which consists of a very large number of particles any two of which repel one another according to a given law of force; the dependence of the equations on the law of force is explicitly given. The equations are those typical of liquids and gases but the analysis suggests how the solid state might arise.

In Chapter 6, the assumptions underlying the theory of dimensional analysis are reviewed and the fundamental Vaschy-Buckingham Pi Theorem is stated. Application is made to the determination of the most general functional forms of the peak values of soil pressure, particle acceleration, velocity and displacement.

CHAPTER 1

GENERAL DISCUSSION OF RESULTS

1.1 PURPOSE OF A THEORETICAL STUDY

The purpose of this project is to develop a theory of wave motion in the ground which will account for phenomena already observed in large scale explosions and will predict phenomena with reasonable accuracy in future explosions. Such a theory should also be useful in suggesting further experiments to increase our knowledge concerning ground waves and their effects on structures, etc.

Reasonably accurate empirical formulas for the variation with distance from an explosion of the peak acceleration, peak pressure, peak transient displacement, etc., in the resulting ground waves were developed by Lampson [7]. The expressions for these quantities in terms of the distance from the explosion were all sums of terms of the form

$$a\lambda^{-n}$$

where a depends on many other quantities (see Chapter 6) and λ is the scaled distance defined by

$$\lambda = r/W^{\frac{1}{3}}$$

in which

r is the distance from the explosion in feet and

W is the equivalent weight of chemical high explosive in pounds.

Although the results of recent H.E. tests held on the Nevada site, as preported by Doll in [4], do not agree in detail with those of Lampson, similar formulas seem to hold. Predictions based on Doll's results and the use of the scaled distance λ were sufficiently accurate, at least, to enable those instrumenting some of the later tests at that site to select instruments capable of recording the data.

In view of the existence of this fairly satisfactory empirical theory, we have conceived of our task as the deeper one of developing a theory of ground as a continuous medium from fundamental principles. This theory would correspond to the theory of hydrodynamics for liquids, gas dynamics for gases, and elasticity or plasticity for solids. The program would then consist in studying the resulting equations to determine first whether the theory agreed with experiment and, if so, to draw further conclusions of interest. In particular, it might be possible to find out what the wave velocity and other similar constants depend on and to determine the meaning of the measurements of earth pressure, etc.

1.2 ORIGINAL FORMULATION OF OUR PROGRAM

Our contract (Project NR 340 040, Contract NONR-222(04)) began on June 15, 1951. The first few months, before receiving any data, were spent by our group in acquainting ourselves with the standard theories of gas dynamics, hydrodynamics, thermodynamics, elasticity, and one-dimensional elastic-plastic flow and with what theories of soil mechanics and experiments on soils could be found in the literature. The material on soils was discouraging: we encountered a great many widely divergent theories and widely differing experimental results. However, the results of Lampson, to which we presently had access, were more encouraging and had a more direct bearing on our problem than had most of the preceding material.

After the survey of relevant background material mentioned above and many discussions of possible important areas of investigation and after a trip by the project director to the Nevada site in September 1951, we decided on the following lines of investigation:

1. An extension to three dimensions of the present one-dimensional theory of flow and wave transmission in an elastic-plastic material.

2. A comparison of the results of this extended theory with the experimental results of Lampson and of the more recent tests.

3. An adaptation of the methods of statistical mechanics to the deduction of appropriate mathematical equations for flow and wave propagation in soils.

4. A study of the meaning of the "pressure" measurements taken in the test.

It was rather evident that the fourth study above would have to await the development of a fairly satisfactory theory of soil mechanics.

1.3 GENERAL DISCUSSION OF RESULTS OBTAINED

Most of the results of our studies are embodied in Chapters 2 to 6. In this section we shall discuss these results in general terms and point out their relations with our general problem. One incomplete result, which is not discussed in any of the following chapters, seems to be of sufficient interest to be included in this section and is discussed under paragraph 1.3.2.

1.3.1 The Elastic Character of the Ground

We believe that results in good agreement with the experimental data will be obtained by assuming that the ground behaves like an elastic solid at distances from the explosion corresponding to values of the scale distance λ which are greater than 4. We have come to believe this so recently that we have not had time to write up the rather difficult analysis in detail.

We were led to consider this assumption after inspection of the data in Doll's report [4] when we noticed that the earth pressures were very small and the peak acceleration (in the first wave) varied like $a\lambda^{-2}$ for distances corresponding to $\lambda > 4$. After a number of unsuccessful studies of the equations of motion in an elastic medium, we found that we could make use of the very important recent work of Professor Pinney on "point source" problems in an elastic half space [13].

One of the problems considered by Professor Pinney in that paper is the determination of the wave motion in an elastic half space generated by the instantaneous injection of a small spherical hole at some point in the half-space. He has determined exact formulas for the resulting displacements on the surface. From this solution one finds that if one inserts this volume in a finite time

according to some law, one can find the resulting continuous wave motion. A rough study of these shows that at sufficiently large distances from the explosion the first wave reaching a station should be propagated beyond with little distortion and that the amplitude should decrease like a λ^{-2} which is in accordance with Doll's observations. Of course the ground is not strictly elastic near the source but the waves sufficiently far out can probably be thought of as having arisen from some equivalent point source in a strictly elastic medium.

1.3.2 The General Stress-Strain Relationship

A first result obtained in connection with the first line of investigation mentioned in §1.2 was the determination of the most general relation between the stress and strain tensors which could be possible in an isotropic medium. These results are set forth in Chapter 2, the only assumption being that the components of the stress tensor in Cartesian coordinates are single-valued functions of the components of the strain tensor only. The object of this study was to prepare to generalize the present one-dimensional theory in which it is assumed that the single component of stress is a fairly general non-linear function of the strain, the function being changed whenever the rate of change of the strain changes sign. However, in trying to carry through the complete extension to three dimensions, the difficulty arose of finding the condition corresponding to the change of sign of the rate of strain, there being six components of strain in the three dimensional case. We were thus led to study other theories of plasticity, such as that of Prager and Hodge [14] in which the stress also depends on the rate of strain.

1.3.3 A Theory of the Mechanics of Soil.

These difficulties inspired Professor Pinney to undertake to develop a theory from first principles. This very interesting theory is presented in Chapter 3. It is very difficult (see below) to check the agreement of any of the known theories with experimental results. If however the effect of the surface of the ground is neglected (i.e. we assume that the explosion takes place very far underground) the equations are greatly simplified. Making this assumption, we found that a first draft of

Professor Pinney's theory yielded a promising agreement with the observed variation of peak acceleration with distance.

1.3.4 An Apirational Solution for Displacements in an Elastic Half-Space

In Chapter 4, Dr. Stoneham has generalized Professor Pinney's point source results to the case where the point source is replaced by a spherical hole of finite size. Although the results are similar to those of Professor Pinney, the methods are somewhat different and constitute a valuable addition to our knowledge concerning the equations of elasticity.

1.3.5 A Report on the Work on Statistical Mechanics.

The method of statistical mechanics has been pursued by Professor Morrey with a view to developing from first principles a theory of ground as a continuous medium. Since the ground is actually composed of small particles the method is not an unnatural one. Several writers on soil mechanics have regarded the ground as consisting of small elastic spherical particles which exert a force upon one another when in contact according to a law developed by Hertz [5]. If this is done and the effects of friction, distortion, and rotation of the particles is neglected, the model obtained reduces to that of a system of point-particles (the centers of the spheres) moving according to a central force law (the force between two particles being zero, of course, when the particles are not in contact). This is a standard model in statistical mechanics. This program has not been completed but some interesting results have already been obtained and many aspects of our method of attack are new. Chapter 5 constitutes a progress report on this work.

In particular, a complete set of equations has been obtained which involve the assumed central force law explicitly. The equations are those appropriate to a non-viscous liquid or gas. However, the analysis gives a strong indication as to how the method can be applied to discuss the solid state. The results also suggest that viscosity is definitely due to the finite size of the particles; viscosity terms can only be found by making a careful study of the approximations made.

1.3.6 On the Application of Dimensional Analysis to Underground Explosions.

This chapter presents an informative and careful study of the conclusions derivable from the considerations of physical dimensions alone.

1.4 IDEAS FOR FUTURE WORK

Our experiences to date suggest that further theoretical work be carried out along the following lines:

1. The completion of the study of Pinney's point source formulas described in paragraph 1.3.1

2. Study of the equations of motion in an elastic half-space with elastic constants varying with depth with a view to the determination of underground effects.

3. A study of the significance of the measurements of earth pressure, assuming the surrounding ground to be elastic.

4. A study of the motion in a combined medium consisting of an elastic half-space with air above it, in order to determine the effects of air blast.

5. A study of Pinney's theory to determine effects nearer to the explosion than the elastic range.

6. A completion of the study of statistical mechanics along the lines presented in Chapter 5 and its extension to include the solid state, mixtures of earth and air, etc., and viscosity effects; this might lead to appropriate equations valid very close to an explosion.

CHAPTER 2

THE GENERAL STRESS-STRAIN RELATIONSHIP FOR AN ISOTROPIC MEDIUM

2.1 INTRODUCTION

In the finite deformation theory of continuous media, as developed by Murnaghan, a general expression for the stress-strain relation in an isotropic medium is given in terms of an elastic potential, whose existence and general functional form is assumed (see [10], pp. 91-94; [16], pp. 314-318). In this chapter, we derive such an expression which is independent of the elastic potential and assumes only the existence of a stress-strain relation given by a continuous function invariant under rotations. The results of this chapter are summed up in Theorems 2.1, 2.2, 2.3, and 2.4.

2.2 SOME BASIC PROPERTIES OF MATRICES

This section is a summary of the matrix notions and summation convention used in this chapter. For an elementary exposition of the details of the results stated here, the reader may consult Sokolnikoff, Tensor Analysis, Chapter 1. Readers familiar with these notions may turn immediately to § 2.3.

A 3×3 matrix A is a set of nine real numbers a_{11} a_{12} a_{13} a_{21} a_{22} a_{23} a_{31} a_{32} a_{33}, called the components of the matrix, which for convenience we write in the form

$$A = \left\| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right\|$$

or, for brevity, write symbolically

$$A = \| a_{\alpha\beta} \|$$

Throughout this chapter we make use of the following

summation convention: if in a term a certain index occurs twice, this is to mean that the term is to be summed with respect to that index from 1 to 3. Thus

$$a_{\alpha\gamma}\, b_{\gamma\beta} = \sum_{\gamma=1}^{3} a_{\alpha\gamma}\, b_{\gamma\beta};\quad c_{\alpha\gamma}\, x^{\gamma} = \sum_{\gamma=1}^{3} c_{\alpha\gamma}\, x^{\gamma}. \qquad (2.1)$$

Given two matrices $A = \|a_{\alpha\beta}\|$ and $B = \|b_{\alpha\beta}\|$, we define their sum $C = A+B$ and their product $D = AB$ by, for every choice of α and β,

$$c_{\alpha\beta} = a_{\alpha\beta} + b_{\alpha\beta};\quad d_{\alpha\beta} = a_{\alpha\gamma}\, b_{\gamma\beta}.$$

Addition and multiplication of matrices obey all the usual rules for the addition and multiplication of real numbers, except that the multiplication of matrices is not commutative.

Equality of matrices is defined as follows:

$$A = B \text{ if for every } \alpha \text{ and } \beta,\quad a_{\alpha\beta} = b_{\alpha\beta}\ .$$

Multiplication of a matrix $A = \|a_{\alpha\beta}\|$ and a number λ is defined by:

$$\lambda A = \|c_{\alpha\beta}\| \text{ where } c_{\alpha\beta} = \lambda\, a_{\alpha\beta}\ .$$

In the usual way we may associate with the matrix $A = \|a_{\alpha\beta}\|$ its determinant

$$|A| = |a_{\alpha\beta}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Determinants obey the simple rule : $|AB| = |A|\,|B|$

Matrices derive their importance from the fact that they are closely related to linear transformations of a space coordinatized by coordinates (y^1, y^2, y^3) into the same space coordinatized by new coordinates (y'^1, y'^2, y'^3). Such linear transformations are defined by the functions:

$$y'^{\alpha} = a_{\alpha\gamma}\, y^{\gamma} \qquad (\alpha = 1,2,3) \qquad (2.2)$$

An especially important matrix is the identity matrix, denoted by I, defined by

$$I = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} = \| \delta_{\alpha\beta} \|$$

where $\delta_{\alpha\beta}$ is the usual Kronecker delta, defined by

$$\delta_{\alpha\beta} = \begin{matrix} 1 & \text{for } \alpha = \beta \\ 0 & \text{for } \alpha \neq \beta \end{matrix}$$

I possesses the property that $IA = AI = A$ for any matrix A.

The identity matrix is an example of a diagonal matrix. A matrix A is diagonal if its components are such that $a_{\alpha\beta} = 0$ for $\alpha \neq \beta$; i.e.

$$A = \begin{Vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{Vmatrix}$$

Diagonal matrices possess two highly useful properties:

$$|A| = a_{11}\ a_{22}\ a_{33}$$

$$AB = \begin{Vmatrix} a_{11}\,b_{11} & 0 & 0 \\ 0 & a_{22}\,b_{22} & 0 \\ 0 & 0 & a_{33}\,b_{33} \end{Vmatrix} = BA$$

if B is also a diagonal matrix, with diagonal components b_{11}, b_{22}, b_{33}.

If to a matrix A, we can find another matrix B such that $AB = I$, then it may be shown that B is unique, and that $BA = I$. We call B the inverse of A, and denote it by A^{-1}. The determinants satisfy $|A^{-1}| = 1/|A|$. In terms

of $A^{-1} = \| a^{-1}_{\alpha\beta} \|$, we could show that the y's of equation 2.2 may be expressed in terms of the y''s by the relation

$$y^{\alpha} = a^{-1}_{\alpha\gamma}\, y'^{\gamma} \quad (\alpha = 1,2,3) \quad . \tag{2.3}$$

Not every matrix possesses an inverse, but every matrix A does possess a transpose $A^{*} = \| a^{*}_{\alpha\beta} \|$ defined by $a^{*}_{\alpha\beta} = a_{\beta\alpha}$. It may be shown $|A^{*}| = |A|$.

A matrix A is said to be symmetric if $a_{\alpha\beta} = a_{\beta\alpha}$ for every α and β, or equivalently if $A^{*} = A$. Notice that an equation involving matrices is equivalent to nine equations involving numbers.

It can be proved that the condition that the linear transformation defined by equation 2.2 correspond to what we mean geometrically by a rotation can be expressed in matrix notation by the conditions

$$A^{*} = A^{-1} \qquad |A| = 1 \tag{2.4}$$

For a rotation we have by equations 2.2 and 2.3, since $a^{-1}_{\alpha\gamma} = a^{*}_{\alpha\gamma} = a_{\gamma\alpha}$,

$$y'^{\alpha} = a_{\alpha\gamma}\, y^{\gamma}\, ; \; y^{\alpha} = a_{\gamma\alpha}\, y'^{\gamma} \tag{2.5}$$

There exists a very useful relation between symmetric matrices, diagonal matrices, and rotations. To every symmetric matrix E, there can be found a rotation C such that $CEC^{-1} = E'$ is a diagonal matrix.

Two matrices E and E' are called similar if they can be transformed into one another by means of linear transformations which possess an inverse, i.e., there exists a matrix C such that $E' = CEC^{-1}$. A numerical valued function of a matrix, f(E), is called an <u>invariant</u> if for similar matrices E and E', f(E) = f(E').

A very important example of an invariant is the characteristic polynomial of a matrix E, defined by

$$\varphi_{\lambda}(E) = |E - \lambda I| \; .$$

If this determinant is expanded in powers of λ , it will be seen to be a cubic polynomial which we write

$$\varphi_{\lambda}(E) = \lambda^{3} - L\lambda^{2} + M\lambda^{2} - N$$

where the coefficients L,M,N are functions of E. For similar matrices E and $E' = CEC^{-1}$, $\varphi_{\lambda}(E) = \varphi_{\lambda}(E')$ since

$$\begin{aligned} |E'-\lambda I| &= |CEC^{-1}-\lambda ICC^{-1}| = |CEC^{-1}-C\lambda IC^{-1}| \\ &= |C(E-\lambda I)C^{-1}| = |C|\,|E-\lambda I|\,|C^{-1}| \\ &= |E-\lambda I| . \end{aligned}$$

Therefore

$$\lambda^{3}- L(E')\lambda^{2}+ M(E')\lambda - N(E') = \lambda^{3}-L(E)\lambda^{2}+M(E)\lambda-N(E)$$

and since these cubic polynomials agree for all λ , the coefficients are equal:

$$L(E') = L(E) \quad M(E') = M(E) \quad N(E') = N(E).$$

L,M, and N are explicitly given by, for $E = \|e_{\alpha\beta}\|$,

$$L(E) = e_{11} + e_{22} + e_{33}$$

$$M(E) = (e_{11}\, e_{22}-e_{12}\, e_{21}) + (e_{22}\, e_{33}-e_{23}\, e_{32}) + (e_{33}\, e_{11}-e_{31}e_{13})$$

$$N(E) = |e_{\alpha\beta}|$$

2.3 THE STRESS-STRAIN FUNCTION

Consider a continuous medium which is undergoing deformation. We suppose the body to be coordinatized by a Cartesian coordinate system. For a given material point P we let (a^1,a^2,a^3) be its coordinates in the initial state and (x^1,x^2,x^3) be its coordinates in the deformed state. We assume that these coordinates are related by differentiable functions:

$$x^{\alpha} = x^{\alpha}(a^1,a^2,a^3) \qquad (\alpha = 1,2,3).$$

The point P has then undergone a displacement $\underline{u}$, due to the deformations, whose components are

$$u^{\alpha} = u^{\alpha}(a^1,a^2,a^3) = x^{\alpha}(a^1,a^2,a^3) - a^{\alpha} \quad (\alpha = 1,2,3). \tag{2.6}$$

As a measure of the strain associated with the deformation in the finite deformation theory we consider the set of nine quantities

$$e_{\alpha\beta} = \frac{1}{2}\left[\frac{\partial u^{\alpha}}{\partial a^{\beta}} + \frac{\partial u^{\beta}}{\partial a^{\alpha}} + \frac{\partial u^{\gamma}}{\partial a^{\alpha}}\frac{\partial u^{\gamma}}{\partial a^{\beta}}\right] \tag{2.7}$$

For the geometric and physical significance of the $e_{\alpha\beta}$ see Sokolnikoff, Mathematical Theory of Elasticity (McGraw Hill, 1946) pp. 28-32. The quantities $e_{\alpha\beta}$ are symmetric, i.e. $e_{\alpha\beta} = e_{\beta\alpha}$, and thus they may be taken as the components of a symmetric matrix E.

Let the Cartesian system (x^1,x^2,x^3) be transformed into a new coordinate system (x'^1,x'^2,x'^3) by a rotation matrix $C = \|c_{\alpha\beta}\|$. Now it may be shown that the quantities $e_{\alpha\beta}$ are the components of a covariant tensor of rank 2 ([10], p. 77; [16], pp. 291-299). This means that if P is a material point at which the strain components in the old and new coordinate systems are denoted by $e_{\alpha\beta}$ and $e'_{\alpha\beta}$ respectively, then

$$\begin{aligned} e'_{\alpha\beta}(x'^1,x'^2,x'^3) &= e_{\gamma\delta}(x^1,x^2,x^3)\ \frac{\partial x^{\gamma}}{\partial x'^{\alpha}}\frac{\partial x^{\delta}}{\partial x'^{\beta}} \\ &= e_{\gamma\delta}(x^1,x^2,x^3)\ c_{\alpha\gamma}\, c_{\beta\delta} \qquad (2.8) \\ &= c_{\alpha\gamma}\, e_{\gamma\delta}(x^1,x^2,x^3)\ c^{-1}_{\delta\beta} \end{aligned}$$

since by equation 2.5

$$\frac{\partial x^{\gamma}}{\partial x'^{\alpha}} = c_{\alpha\gamma} = c^{*}_{\gamma\alpha} = c^{-1}_{\gamma\alpha}\,.$$

The reader may check equation 2.8 for himself by substituting in equation 2.7

$$\frac{\partial u'^{\alpha}}{\partial a'^{\beta}} = c_{\alpha\gamma}\,\frac{\partial u^{\gamma}}{\partial a^{\delta}}\,c_{\beta\delta}\,.$$

We have shown then that if $E' = \|e'_{\alpha\beta}\|$ is the matrix of

the strains in the new coordinate system, the following relation holds between the strains in two coordinate systems which may be transformed into one another by a rotation C:

$$E' = CEC^{-1}. \tag{2.9}$$

Next consider the set of nine quantities $\tau_{\alpha\beta}(\alpha,\beta = 1,2,3)$ associated with the (x^1,x^2,x^3) coordinate system which it is known are sufficient to characterize the state of stress at any point of the medium. Similarly let $\tau'_{\alpha\beta}$ be the corresponding set of quantities associated with the (x'^1,x'^2,x^3) coordinate system. The $\tau_{\alpha\beta}$ may be regarded as the components of both a matrix and a tensor, and we may obtain an equation for the $\tau_{\alpha\beta}$ similar to equations 2.8 and 2.9 ([15], pp. 44-45):

$$\tau'_{\alpha\beta} = c_{\alpha\delta} \quad \tau_{\alpha\beta} \quad c^{-1}_{\beta\delta} \tag{2.10}$$

$$T' = C \qquad T \qquad C^{-1} \tag{2.11}$$

where $T = \|\tau_{\alpha\beta}\|$ is the stress matrix.

Now let us suppose that there is a relationship between the stress and strain matrices, defined by means of a continuous matrix function F as follows:

$$T = F(E) \tag{2.12}$$

By this we mean that every component $\tau_{\alpha\beta}$ of T is a function of the six independent components of E:

$$\tau_{\alpha\beta} = f_{\alpha\beta}(E) = f_{\alpha\beta}(e_{11},e_{22},e_{33},e_{12},e_{23},e_{31}) \tag{2.13}$$

To say that F is a continuous matrix function is to say that each $f_{\alpha\beta}$ is a continuous function of its six arguments. If the medium is assumed to be isotropic, then the stress-strain relation must be invariant under rotation ([15], p. 65). That is, let P be a point in the medium whose coordinates in a given coordinate system are (x^1,x^2,x^3), and are (x'^1,x'^2,x'^3) in a new coordinate system obtained from the given one by a rotation C. At P, let the strain and stress matrices in the given coordinate system be denoted by E and T respectively, and in the new system by E' and T'

respectively. Then

$$T' = F(E') \tag{2.14}$$

But by equations 2.11 and 2.9

$$T' = CTC^{-1} = CF(E)C^{-1} = CF(C^{-1}E'C)C^{-1} \tag{2.15}$$

so that equating equations 2.14 and 2.15

$$C^{-1}F(E')C = F(C^{-1}E'C).$$

We have thus shown the

Theorem 2.1 The matrix function specifying the stress-strain relation (of equation 2.12) in an isotropic medium must be such that, for any symmetric matrix E and any rotation C,

$$F(C^{-1}EC) = C^{-1}F(E)C. \tag{2.16}$$

The remainder of this chapter will deal with the problem of characterizing a matrix function of this kind.

2.4 THE STRESS-STRAIN FUNCTION FOR DIAGONAL MATRICES

We note first that F is completely determined by its values for diagonal matrices. For to every symmetric matrix E, there may be found a rotation C such that $CEC^{-1} = D$ is a diagonal matrix. Then by equation 2.16, $F(E) = CF(D)C^{-1}$, which establishes the remark made.

However, since for a given E, the rotation C may be followed by another which carries a first diagonal matrix into another with its elements permuted, the function F for diagonal matrices must satisfy certain conditions of symmetry.

For a diagonal matrix

$$E = \begin{Vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{Vmatrix}$$

we have by equations 2.12 and 2.13

$$F(E) = \left\| f_{\alpha\beta}(a,b,c) \right\|$$

If

$$C = \left\| \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} \right\| \qquad C^{-1} = \left\| \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \right\|$$

then

$$CEC^{-1} = \left\| \begin{matrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & a \end{matrix} \right\| \qquad CF(E)C^{-1} = \left\| \begin{matrix} f_{22} & f_{32} & f_{12} \\ f_{23} & f_{33} & f_{13} \\ f_{21} & f_{31} & f_{11} \end{matrix} \right\| \tag{2.17}$$

where $f_{\alpha\beta}$ stand for $f_{\alpha\beta}(a,b,c)$. Forming $F(CEC^{-1})$, we equate it, component by component, to $CF(E)C^{-1}$. Among the relations we obtain are

$$f_{11}(b,c,a) = f_{22}(a,b,c) \text{ so } f_{22}(a,b,c) = f_{11}(b,c,a) \tag{2.18}$$

$$f_{22}(b,c,a) = f_{33}(a,b,c) \text{ so } f_{33}(a,b,c) = f_{22}(b,c,a) = f_{11}(c,a,b)$$

$$f_{13}(b,c,a) = f_{12}(a,b,c) \text{ so } f_{13}(a,b,c) = f_{12}(c,a,b)$$

$$f_{23}(b,c,a) = f_{13}(a,b,c) \text{ so } f_{23}(a,b,c) = f_{13}(c,a,b) = f_{12}(b,c,a)$$

$$f_{21}(b,c,a) = f_{23}(a,b,c) \text{ so } f_{21}(a,b,c) = f_{23}(c,a,b) = f_{12}(a,b,c)$$

$$f_{31}(b,c,a) = f_{21}(a,b,c) \text{ so } f_{31}(a,b,c) = f_{21}(c,a,b) = f_{12}(c,a,b)$$

$$f_{32}(b,c,a) = f_{31}(a,b,c) \text{ so } f_{32}(a,b,c) = f_{31}(c,a,b) = f_{12}(b,c,a).$$

Thus in terms of f_{11} and f_{12} the function F may be represented

$$\left\| f_{\alpha\beta}(a,b,c) \right\| = \left\| \begin{matrix} f_{11}(a,b,c) & f_{12}(a,b,c) & f_{12}(c,a,b) \\ f_{12}(a,b,c) & f_{11}(b,c,a) & f_{12}(b,c,a) \\ f_{12}(c,a,b) & f_{12}(b,c,a) & f_{11}(c,a,b) \end{matrix} \right\|$$

Next, if

$$C = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{Vmatrix} \qquad C^{-1} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{Vmatrix}$$

then

$$CEC^{-1} = \begin{Vmatrix} a & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & b \end{Vmatrix} \qquad CF(E)C^{-1} = \begin{Vmatrix} f_{11} & f_{13} & -f_{12} \\ f_{13} & f_{33} & -f_{23} \\ -f_{12} & -f_{23} & f_{22} \end{Vmatrix} \quad (2.19)$$

In the same way that equation 2.18 was obtained from equation 2.17, we obtain from equation 2.19

$$f_{11}(a,c,b) = f_{11}(a,b,c)$$

$$f_{12}(a,c,b) = f_{13}(a,b,c) \text{ so } f_{13}(a,c,b) = f_{12}(a,b,c)$$

$$f_{13}(a,c,b) = -f_{12}(a,b,c).$$

We therefore infer about f_{12} that

$$f_{12}(a,b,c) = -f_{12}(a,b,c),$$

so that $f_{12} \equiv 0$, and infer about f_{11} that

$$f_{11}(a,b,c) = f_{11}(a,c,b).$$

Application of other specific rotations gives no new information. If we write f instead of f_{11} we have proved

Theorem 2.2 If $F(E)$ is a matrix function of a symmetric matrix E such that, for any rotation C,

$$F(C^{-1}EC) = C^{-1}F(E)C,$$

then for diagonal matrices

$$F\left[\begin{Vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{Vmatrix}\right] = \begin{Vmatrix} f(a,b,c) & 0 & 0 \\ 0 & f(b,c,a) & 0 \\ 0 & 0 & f(c,a,b) \end{Vmatrix} \qquad (2.20)$$

where f is a function symmetric in the last two arguments:

$$f(a,b,c) = f(a,c,b) \qquad (2.21)$$

2.5 A NECESSARY AND SUFFICIENT CONDITION FOR THE STRESS-STRAIN FUNCTION

Lemma 2.1 Suppose $f(a,b,c)$ is any polynomial of degree n which is symmetric in b and c. Then there exists polynomials $f_i(a,b,c)$ of degree less than or equal to i which are symmetric in (a,b,c) such that

$$f(a,b,c) = \sum_{i=0}^{n} f_i(a,b,c)a^{n-i}.$$

This is a special case of the theorem given in [3], p. 132. In fact, since $f(a,b,c)$ is a symmetric polynomial in b and c, we may write

$$f(a,b,c) = P(a,b+c,bc)$$

where P is a polynomial in these three variables [3], p. 129. We note that

$$b+c = L-a$$
$$bc = a(a-L)+M,$$

where

$$L = a+b+c$$
$$M = ab+ac+bc.$$

Hence

$$f(a,b,c) = P(a,L-a,a(a-L)+M).$$

By rearranging in the explicit powers of a we obtain

$$f(a,b,c) = \sum_{i=0}^{n} g_i(L,M)a^{n-i} \qquad \text{Q.E.D.}$$

Applying this Lemma we prove the following

Theorem 2.3 Suppose the symmetric matrix T is a function F(E) of the symmetric matrix E, where the components of T are polynomials in the components of E, and such that the relation

$$CF(E)C^{-1} = F(CEC^{-1}) \tag{2.22}$$

holds for all rotation matrices C. Then there are polynomials $\varphi_0(L,M,N)$, $\varphi_1(L,M,N)$, $\varphi_2(L,M,N)$ in the invariants L,M,N such that

$$F(E) = \varphi_0(L,M,N)I + \varphi_1(L,M,N)E + \varphi_2(L,M,N)E^2 \tag{2.23}$$

Proof: Suppose first that E is the diagonal matrix of Theorem 2.2, then by Theorem 2.2 and Lemma 2.1

$$F\left[\begin{Vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{Vmatrix}\right] = \sum_{i=0}^{n} f_i(a,b,c) \begin{Vmatrix} a^{n-i} & 0 & 0 \\ 0 & b^{n-i} & 0 \\ 0 & 0 & c^{n-i} \end{Vmatrix} \tag{2.24}$$

where each $f_i(a,b,c)$ is a symmetric polynomial in a, b, c. It is well known [3], p. 129 that each f_i can be written in the form

$$f_i(a,b,c) = \Psi_i(L,M,N)$$

where

$$L = a+b+c, \quad M = ab+ac+bc, \quad N = abc$$

and where Ψ_i is a polynomial in L,M,N. Also, from the definition of matrix products

$$\begin{Vmatrix} a^{n-i} & 0 & 0 \\ 0 & b^{n-i} & 0 \\ 0 & 0 & c^{n-i} \end{Vmatrix} = \begin{Vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{Vmatrix}^{n-i}$$

Thus, from equation 2.24

$$F(E) = \sum_{i=0}^{n} \psi_i(L,M,N)E^{n-i} \tag{2.25}$$

One verifies directly that the diagonal matrix E satisfies the relation

$$E^3 - LE^2 + ME - NI = 0$$

so that, we may express all the powers of E beyond the second in terms of I, E, and E^2. Making this substitution in equation 2.25, we obtain for diagonal matrices E, the relation

$$F(E) = \varphi_0(L,M,N)I + \varphi_1(L,M,N)E + \varphi_2(L,M,N)E^2 \tag{2.26}$$

in which φ_0, φ_1, φ_2 are polynomials.

But now, suppose E is any symmetric matrix. Let C be a rotation matrix such that $CE'C^{-1} = E$, E' being diagonal. Let L',M',N' be the invariants for E', then by § 2.2

$$L = L', \quad M = M', \quad N = N' . \tag{2.27}$$

Also we note that

$$CIC^{-1} = I, \quad CE'C^{-1} = E, \quad C(E')^2C^{-1} = (CE'C^{-1})(CE'C^{-1}) = E^2 , \tag{2.28}$$

Hence we obtain the equation 2.26 for F(E) in general.

If in Theorem 2.3 we replace the word "polynomial", wherever it occurs, by the expression "twice differentiable and continuous function" then the theorem may be shown to still hold. In its present form, the proof is somewhat tedious. It proceeds by uniformly approximating F(E) by polynomial relations $F_n(E)$ for which by Theorem 2.3

$$F_n(E) = \overset{n}{\varphi}_0(LMN)I + \overset{n}{\varphi}_1(LMN)E + \overset{n}{\varphi}_0(LMN)E^2 . \tag{2.29}$$

One then shows that the polynomials $\overset{n}{\varphi}_0$, $\overset{n}{\varphi}_1$, $\overset{n}{\varphi}_2$ converge to continuous functions φ_0, φ_1, φ_2, and equation 2.23 is established. It is hoped that in a later report this

work may be simplified and presented.

Theorem 2.3 admits the following converse, which follows immediately from equations 2.27 and 2.28:

Theorem 2.4 If $F(E)$ is a relation of the type in Theorem 2.3 equation 2.23, then it satisfies the relation

$$F(CEC^{-1}) = CF(E)C^{-1}$$

for all symmetric E and rotation matrices C.

CHAPTER 3

A THEORY OF THE MECHANICS OF SOIL

3.1 INTRODUCTION

Static and dynamic stress distributions in soils are important in geophysics, in civil engineering, and in certain other fields. Typical problems occur in the theories of building foundations, in earthquake theory, in seismic explosions, and in the resistance of structures to earthquake motion. In seismological theories of earthquake waves, the earth is usually approximated as an elastic solid. While this approximation is probably good at depth it is questionable near the surface where the pressure is not great. Moreover, as indicated by the Rayleigh wave phenomena, the surface effects are especially important in geophysics. The purpose of this paper is to develop a theory which may more closely approach the mechanics of soils than the classical theory of elasticity.

It is, of course, hopeless to expect to determine the individual behavior of each soil particle, and no useful theory can involve such detailed knowledge. Our basic data must be taken as averages over many particles, and we are limited to no more specific predictions than those of the average behavior of many particles. Roughly speaking we will be concerned with distances of three orders of magnitude. Distances of the order of the dimensions of the soil particles or less may be called microscopic. Distances of the order of the dimensions of the whole soil field may be called macroscopic. Distances large enough that under uniform conditions, stresses averaged over areas of these dimensions have suitably small standard deviations may be called mesoscopic. For example, in certain soils, distances of the order of hundredths of an inch might be microscopic, distances of the order of inches might be mesoscopic, and distances greater than ten feet might be regarded as macroscopic.

Experimental data will be mesoscopic because detection instruments are mesoscopic. Our predictions will apply to at least mesoscopic dimensions. The stresses we will discuss will be forces applied to at least mesoscopic areas, unlike stresses in the theory of continuous media, where the areas of application are allowed to tend to zero, thus defining the stresses as point functions. Actually we will also

speak of stresses as point functions, but will really mean average stresses averaged over some mesoscopic area centered at the point in question. Of course these may differ widely from the actual stresses at that point.

It will be assumed that the soil is homogeneous in the mesoscopic sense. By this we mean that the particle arrangements and particle size distributions, etc., obtained from a series of samples of mesoscopic dimension are the same within suitably small deviations. The assumption of isotropy in a similar mesoscopic sense will considerably simplify the theory, although a non-isotropic theory, similar to the elastic theory of crystals might be developed. Mesoscopic isotropy will be assumed in the present theory.

The hypothetical material discussed here differs from an elastic solid in two particulars. First, it is assumed that the material cannot support tension. Second, a Coulomb friction mechanism governing the internal particles is postulated in the plastic yield condition equation 3.13 and in the assumed form of the frictional loss of internal energy.

Heuristic arguments for the assumed forms are given, based on certain physical considerations. However it must be borne in mind that at this point adequate experimental evidence does not exist as to the relative importance of various plausible theoretical mechanisms in the mechanics of soils. All such theories as this are therefore tentative; one is justified in developing them as long as serious conflicts between theory and experiments do not arise to diminish their plausibility.

To save having to rewrite a number of formulas and to avoid breaking into the main line of argument, certain corrections to take into account the effects of water in the soil are developed before the theory of dry soil is developed. The arguments employed are very rough--one might say semi-plausible. The final results do not seem unreasonable. Further experimental guidance is needed.

In § 3.2 a rough analysis of the effects of surface tension in the soil water is given. A similar treatment of the effect of water viscosity is given in §3.3. In §3.4, the condition for plastic yielding of the soil is derived.

3.2 WET SOIL - SURFACE TENSION

We will now suppose that a sufficiently large amount of water has entered the soil to fill an appreciable fraction of the air spaces between particles in a uniform manner throughout the region of interest. Under these circumstances the connected bodies of water in the soil may be expected to be of very much larger than microscopic dimensions--possibly of macroscopic dimensions, for the cohesion of water tends to keep these bodies from breaking up. In such water bodies the pressure due to surface tension will be constant. It seems not unreasonable to assume this pressure to be constant throughout the medium.

The pressure due to surface tension is given by

$$p = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right) , \qquad (3.1)$$

where R_1 and R_2 are the principal radii of curvature of the water surface, and are considered positive or negative according as the corresponding centers of curvature lie on the water side or the air side of the surface, respectively. T is the constant of surface tension and has the value 73 dynes/cm for water at 20°C.

Consider the water near a point of contact C between two particles. The principal curvatures of the water surface will be in planes roughly parallel to the plane of contact, and perpendicular to this plane. The radius of curvature, R_1, in the latter plane will generally be small compared to the other principal radius of curvature R_2, so by equation 3.1, R_1 is nearly constant. R_1 is roughly proportional to the distance between the two particles at the water surface, and this is roughly proportional to the area of the cross-section of the water in the plane of contact which is therefore roughly constant for all particles large enough.

Since p is constant it would appear that each sufficiently large particle is subjected to roughly the same normal force in the region around each point of contact, this force being roughly independent of the amount of water present. Smaller particles would be completely inundated by water, but in this case they may be considered to be joined to their neighbors into a larger composite "particle" which will be of type already discussed.

Statistically we may expect the individual particles

to have numbers of points of contact with other particles roughly proportional to their surface area. Therefore in a rough way, the forces due to surface tension acting on the particles are statistically proportional to their cross-sections. Therefore these forces per unit cross-section area are statistically constant, and the effect of surface tension is, according to these assumptions, to add a constant pressure to the stress in the medium which is at most weakly dependent on the degree of saturation of the soil by the water. Applying dimensional reasoning to equation 3.1, we might expect this pressure to be given by

$$p = 4T/d \tag{3.2}$$

where d is some sort of statistically derived distance measurement, such as a mean soil grain diameter or something similar.

If there is very little water present, or if the soil is virtually saturated, these conclusions may not, of course, be expected to hold. On the other hand surface tension probably plays a minor role in these cases.

In practice p may be small. By equation 3.2, for $d = 0.1$ mm, $p = 0.42$ lb/in^2.

3.3 WET SOIL - VISCOSITY

When the particles of the soil move with respect to one another, the viscosity of the water may play a role. To a lesser extent the elastic deformation of the particles causes motion in the water even when there is no slipping, and this motion may generate viscous forces. We shall attempt a crude analysis of the effect of viscosity in the water on the mesoscopic components of stress.

Consider the liquid between two particles in contact. When relative motion occurs it must be expected that the liquid in the immediate vicinity of the points of contact slips over the solid surface in a semi-solid manner, possibly exhibiting marked internal turbulence. We will not pretend to analyse the behavior of the fluid in this region, but will assume that its effect can be accounted for by modifying the Coulomb coefficient of friction of the medium.

Outside this "Coulomb friction" region we will assume that the water flows as a classical viscous fluid. The shearing motion and therefore the viscous effect is greatest where the particle surfaces are nearest together. Suppose

we consider the flow to be analogous to that between two parallel plates when one is moving edgewise with a velocity V relative to the other. In this case each plate suffers a shearing stress $\mu V/z$, where z is the distance between the plates and μ is the coefficient of viscosity of the water.

We might use the expression $\mu V/z$ for the viscous drag in this case where z is variable. This is not a very good approximation, but it is enormously more convenient than an attempt to make a very accurate analysis. This expression probably overestimates the drag where the surfaces are farther apart and less nearly parallel.

We will consider the case of a sphere in contact with a plane, and will attempt to extrapolate from this case to the general one.

Let the plane in question be the $z = 0$ plane in an x,y,z - rectangular coordinate system, and let the sphere be of radius R and tangent to this plane at the origin. Suppose the "Coulomb friction" region is a small circle of radius a about the origin, lying in the $z = 0$ plane. Near the origin the distance from the sphere to the plane is $r^2/(2R) + O(r^4/R^3)$, where r is the distance from the z-axis to the point in question. Accordingly we will approximate the stress by

$$2\mu VR/r^2 \left[1+C(r^2/R^2)\right].$$

Integrating this from the Coulomb friction region to a circle of radius R' about the origin, the total force is

$$F = 4\pi\mu VR \ \ln(R'/a) \left[1+O(R'^2/R^2)\right]$$

In general in a soil we will be concerned with the rate of strain $\dot{e} = V/R$ rather than the velocity V of one particle with respect to another. Then dividing F by the area πR^2 of a great circle cross-section of the sphere,

$$\tau = 4\mu\dot{e} \ \ln(R'/a) \left[1+O(R'^2/R^2)\right] \tag{3.3}$$

represents a shearing force per unit area of the sphere's projection on the plane due to the viscosity of the water.

The quantity R' used in this analysis will be taken near to the radius of the water surface if the sphere is not immersed, or of the order of the radius of the sphere

if it is. It will be assumed that $a << R$. If R' is not large with respect to a it may be assumed that very little water is present in the medium and that its viscous effects will not be important. On the other hand if $R' >> a$, the function $\ln(R'/a)$ is a slowly varying function of its argument. Both the effect of the $O(R'^2/R^2)$ term and a correction for the error of our parallel plate flow approximation would tend to make the coefficient of $4\mu\dot{e}$ in equation 3.3 even more slowly varying than this. We will accordingly make the extrapolation that in general the viscous shear is given by an expression of the form

$$\tau = M\dot{e}, \tag{3.4}$$

where M is a constant of the medium.

The meaning of this is that we will consider the viscosity of water in the soil to be accounted for simply by adding to our mesoscopic components of stress the usual expressions due to viscosity in a continuous liquid whose coefficient of viscosity is M.

M will be a constant for a mesoscopically homogeneous and isotropic medium. However it will differ from one such medium to another. It may be expected to depend on the distribution of particle sizes in the medium and (although not strongly) on the degree of saturation of the soil.

3.4 THE COULOMB FRICTION YIELD CONDITION

Dry soil is incapable of supporting appreciable tension. Accordingly the three principal stresses will be assumed to be compressive and, according to convention, will be non-positive. If one of these should become zero at an interior point of the medium, the soil will break apart, initiating a new regime.

Assume now that all principal stresses are negative. Then any mesoscopic surface area supports a non-zero normal stress. A soil differs from an elastic solid in that it can sustain only a limited shearing stress on this surface area without suffering permanent or plastic deformation. We will assume that the mechanism of yielding and deformation are similar in nature to those of Coulomb friction and will use physical arguments based on the idea of Coulomb friction in justification of the yielding hypothesis equation 3.13 advanced.

At any given point in the soil the ratio R of the

tangential to the normal stress can be calculated with respect to a mesoscopic surface area oriented in any direction. By varying the direction, R may be made to take on its maximum value R_m. We will assume that as long as R_m remains less than a certain constant k (called the coefficient of friction for the soil), sliding does not occur, and the system is conservative in the immediate neighborhood of the point in question. On the other hand, if R_m grows as large as k, sliding will occur, with attendant loss of energy by friction, in such a manner as to keep R_m from growing larger than k. Therefore we have

$$R_m \leq k \tag{3.5}$$

k may not be equal to the coefficient of friction between any particular pair of particles, which may vary considerably from one pair to another if several different materials are present in the soil. It is a mesoscopic rather than a microscopic parameter.

We will now calculate R_m. Let τ_1, τ_2, τ_3 be the principal stresses at the point P of interest, and assume $\tau_1 < \tau_2 < \tau_3 < 0$. Establish a rectangular coordinate system with origin at P and axes along the principal directions of stress at P. Now consider a plane through P whose normal has direction cosines ν_1, ν_2, ν_3 with respect to this system. From the tensor law of transformation of stress, the component of force normal to this surface is

$$\tau_1 \nu_1^2 + \tau_2 \nu_2^2 + \tau_3 \nu_3^2 ,$$

and the square of the magnitude of the total force is

$$\tau_1^2 \nu_1^2 + \tau_2^2 \nu_2^2 + \tau_3^2 \nu_3^2 .$$

Therefore

$$R^2 + 1 = \frac{\tau_1^2 \nu_1^2 + \tau_2^2 \nu_2^2 + \tau_3 \nu_3^2}{(\tau_1 \nu_1^2 + \tau_2 \nu_2^2 + \tau_3 \nu_3^2)^2} \tag{3.6}$$

Moreover

$$\nu_1^2 + \nu_2^2 + \nu_3^2 = 1 \tag{3.7}$$

From equation 3.6, equation 3.7 we may write

$$R^2 + 1 = \frac{Ax+By+1}{(ax+by+1)^2} , \tag{3.8}$$

where $x = \nu_1^2$, $y = \nu_2^2$, $A = \tau_1^2/\tau_3^2 - 1$, $B = \tau_2^2/\tau_3^2 - 1$, $a = \tau_1/\tau_3 - 1$, $b = \tau_2/\tau_3 - 1$. We wish to maximize this expression as a function of x and y subject to the restrictions

$$x \geqslant 0, \quad y \geqslant 0, \quad x + y \leqslant 1 \tag{3.9}$$

It is easily seen that $\partial R/\partial x$ and $\partial R/\partial y$ cannot vanish simultaneously except when $\tau_1 = \tau_2$, a limiting case we are not now considering. The maximum value of R must therefore correspond to (x,y) on one of the boundary lines of the region equation 3.9 in the x,y-plane.

Suppose $x = 0$. Then by equation 3.8, $\partial R/\partial y = 0$ if

$$B(by+1) - 2b(By+1) = 0,$$

giving $y = (B-2b)/(Bb) = \tau_3/(\tau_3 + \tau_3)$, and, by equation 3.8, $R = |\tau_2 - \tau_3|/\sqrt{(4\,\tau_2\tau_3)}$.

Similarly the maximum value of R corresponding to $y = 0$ is $R = |\tau_1 - \tau_3|/\sqrt{(4\,\tau_1\,\tau_3)}$.

Finally consider the boundary $x + y = 1$. By equation 3.8,

$$R^2 + 1 = \frac{(A-B)x+B+1}{\left[(a-b)x+b+1\right]^2}$$

Then $\partial R/\partial x = 0$ if

$$(A-B)\left[(a-b)x+b+1\right] - 2(a-b)\left[(A-B)x+B+1\right] = 0,$$

giving

$$x = \left[(A-B)(b+1)-2(a-b)(B+1)\right] \Big/ \left[(a-b)(A-B)\right]$$

$$= \tau_2/(\tau_1 + \tau_2),$$

and $R = |\tau_1 - \tau_2|/\sqrt{(4\,\tau_1\,\tau_2)}$.

These three expressions for R have the same form;

the greatest is that in which the two variables are farthest apart. Therefore $R_m = |\tau_1 - _3| / \surd (4 \tau_1 \tau_3)$, so be equation 3.5

$$\frac{1-K}{1+K} \leq \frac{\tau_3}{\tau_1} \leq 1,$$

where

$$K \equiv \cos(2\chi) = k/\surd(1+k^2). \qquad (3.10)$$

For wet soil, the surface tension correction of § 3.2 may be incorporated, giving, for p a non-negative constant,

$$\tan^2\chi = \frac{1-K}{1+K} \leq \frac{p-\tau_3}{p-\tau_1} \leq 1. \qquad (3.11)$$

The limiting cases $\tau_1 = \tau_2$ and $\tau_2 = \tau_3$ give results in agreement with this.

When

$$\tan^2\chi = \frac{1-K}{1+K} < \frac{p-\tau_3}{p-\tau_1} \leq 1, \qquad (3.12)$$

the system is conservative, but when

$$\frac{p-\tau_3}{p-\tau_1} = \frac{1-K}{1+K} = \tan^2\chi, \qquad (3.13)$$

Coulomb frictional yielding may occur with attendant energy loss. This condition is assumed always to hold during yielding, although it may also hold in the transition between the elastic state and the yielding state but just before yielding takes place.

3.5 THE MECHANICAL EQUATIONS

We will now derive the fundamental mechanical equations for our soil model, following the method of Murnaghan with appropriate modifications.

It will be necessary to study the medium in both the deformed and undeformed condition. A rectangular Lagrangian coordinate system with coordinates a^1, a^2, a^3 will describe the material points of the medium in the undeformed state, and a rectangular Eulerian coordinate system with coordinates x^1, x^2, x^3 will describe the material points of the medium in the deformed state. The Lagrangian system describes events with respect to the medium, and the Eulerian system describes events with respect to a fixed reference system.

We use mesoscopic stresses, as defined in §3.1; otherwise the definition of stress is the familiar one. That is, τ^{ij} denotes the component of force in the direction of the negative x^i-axis exerted per unit surface normal to the x^j-axis by the material lying on the side of this surface corresponding to lower values of the coordinate x^j. In this case, however, the surface element must be of mesoscopic dimensions.

For brevity we denote $\partial A/\partial x^\sigma$ by $A_{,\sigma}$

In the strained state consider any volume V of mesoscopic dimensions surrounded by a surface S. Let δx^1, δx^2, δx^3 be virtual displacements of the points of V as measured in the x-system.

The virtual work due to the surface stresses over S is

$$\int_S (\pm\,\delta_{\alpha\beta}\tau^{\alpha 1}\,\delta x^\beta dx^2 dx^3 \pm \delta_{\alpha\beta}\tau^{\alpha 2}\,\delta x^\beta dx^3 dx^1 \pm \delta_{\alpha\beta}\tau^{\alpha 3} dx^1 dx^2)$$

$$= \int_V \Big[(\delta_{\alpha\beta}\tau^{\alpha 1}\,\delta x^\beta)_{,1} + (\delta_{\alpha\beta}\tau^{\alpha 2}\,\delta x^\beta)_{,2} + (\delta_{\alpha\beta}\tau^{\alpha 3}\,\delta x^\beta)_{,3}\Big] dx^1 dx^2 dx^3 \qquad (3.14)$$

$$= \int_V (\delta_{\alpha\beta}\,\tau^{\alpha\sigma}\,\delta x^\beta)_{,\sigma}\, dV = \int_V (\tau^{\alpha\sigma}\,\delta x_\alpha)_{,\sigma}\, dV,$$

where the summation convention is used and where

$$\delta x_i = \delta_{i\sigma}\,\delta x^\sigma \qquad (3.15)$$

If F^1, F^2, F^3 are components of external force per unit mass, the corresponding virtual work is

$$\int \rho\, F^\alpha\,\delta x_\alpha\, dV, \qquad (3.16)$$

ρ being the density of the medium.

Finally let δU denote the variation in internal energy per unit mass corresponding to the variations δx^1, δx^2, δx^3. By the Principle of Virtual Work,

$$\int_V \Big[(\tau^{\alpha\sigma}\,\delta x_\alpha)_{,\sigma} + \rho\, F^\alpha\,\delta x_\alpha - \rho\,\delta U\Big] dV = 0,$$

or

$$\int_V \left[(\tau^{a\sigma}_{,\sigma} + \rho F^a)\, \delta x_a + \tau^{a\sigma} (\delta x_a)_{,\sigma} - \rho \delta U \right] dV = 0 \quad (3.17)$$

Rigid virtual displacements of the medium are characterized by

$$(\delta x_i)_{,j} + (\delta x_j)_{,i} = 0. \quad (3.18)$$

In particular, translations are characterized by

$$(\delta x_i)_{,j} = 0. \quad (3.19)$$

Under a rigid displacement of the medium the internal energy is unchanged, so $\delta U = 0$. By equation 3.17, equation 3.19, since δx_a is arbitrary,

$$\tau^{i\sigma}_{,\sigma} + \rho F^i = 0. \quad (3.20)$$

By equation 3.17 for a rigid displacement, using equation 3.20

$$\int_V (\tau^{a\beta} - \tau^{\beta a})(\delta x_a)_{,\beta}\, dV + \int_V \tau^{a\beta} \left[(\delta x_a)_{,\beta} + (\delta x_\beta)_{,a} \right] dV = 0$$

By equation 3.18, the right-hand integral vanishes. Since $(\delta x_a)_{,\beta}$ is arbitrary we must have

$$\tau^{ij} = \tau^{ji} . \quad (3.21)$$

That is, the stress tensor is symmetric, as usual.

Equation 3.17 may now be written

$$\int_V \left[\tau^{a\beta} (\delta x_a)_{,\beta} - \rho \delta U \right] dV = 0. \quad (3.22)$$

3.6 THE INTERNAL ENERGY U

The virtual change δU in the internal energy is given by

$$\delta U = \delta W + \delta W' + \delta W'', \quad (3.23)$$

where δW is the change in elastic energy, $\delta W'$ is the change in energy due to viscosity in the water, and $\delta W''$ is the change in energy due to Coulomb friction loss.

W is the elastic energy of the body when distorted,

and depends on the degree and manner of distortion of the body. A measure of this distortion is obtained by comparing the differential of distance in the strained and unstrained states.

Let ds_o and ds denote the differentials of length in the undistorted and in the distorted states respectively. Then

$$ds_o^2 = \delta_{\sigma\tau}\, da^\sigma da^\tau , \quad ds^2 = \delta_{\alpha\beta} dx^\alpha dx^\beta , \qquad (3.24)$$

or

$$ds_o^2 = \delta_{\sigma\tau}\, a^\sigma_{,\alpha} a^\tau_{,\beta}\, dx^\alpha dx^\beta , \quad ds^2 = \delta_{\alpha\beta} x^\alpha_{,\sigma} x^\beta_{,\tau}\, da^\sigma da^\tau . \qquad (3.25)$$

Therefore

$$ds^2 - ds_o^2 = 2\,\epsilon_{\alpha\beta} dx^\alpha dx^\beta = 2\,\eta_{\sigma\tau}\, da^\sigma da^\tau , \qquad (3.26)$$

where

$$\epsilon_{ij} = \tfrac{1}{2}(\delta_{ij} - \delta_{\sigma\tau}\, a^\sigma_{,i} a^\tau_{,j}), \qquad (3.27)$$

$$\eta_{ij} = \tfrac{1}{2}\left(\delta_{\alpha\beta} \frac{\partial x^\alpha}{\partial a^i} \frac{\partial x^\beta}{\partial a^j} - \delta_{ij}\right) \qquad (3.28)$$

By equation 3.26

$$\epsilon_{ij} = \eta_{\sigma\tau}\, a^\sigma_{,i} a^\tau_{,j} ,$$

$$\eta_{ij} = \epsilon_{\alpha\beta} \frac{\partial x^\alpha}{\partial a^i} \frac{\partial x^\beta}{\partial a^j} . \qquad (3.29)$$

The quantities ϵ_{ij} and η_{ij} measure the distortion of the body and are components of the Eulerian and Lagrangian <u>strain tensors</u> respectively. W may be expressed as a function of the three Eulerian strain invariants I_1, I_2, I_3, or the three Lagrangian strain invariants J_1, J_2, J_3, where

$$|\epsilon_{ij} + \kappa\,\delta_{ij}| = \kappa^3 + I_1\kappa^2 + I_2\kappa + I_3, \qquad (3.30)$$

$$|\eta_{ij} + \kappa\,\delta_{ij}| = \kappa^3 + J_1\kappa^2 + J_2\kappa + J_3. \qquad (3.31)$$

These invariants are simply related. By equation 3.27, equation 3.28,

$$\left|\eta_{ij} + \kappa\,\delta_{ij}\right| = \frac{1}{8}\left|\delta_{\alpha\beta}\frac{\partial x^\alpha}{\partial a^i}\frac{\partial x^\beta}{\partial a^j} - (1-2\kappa)\,\delta_{ij}\right|$$

$$= \frac{1}{8}\left|\delta_{\alpha\beta} - (1-2\kappa)\,\delta_{ij}a^i_{,\alpha}a^j_{,\beta}\right|\left|\frac{\partial x^r}{\partial a^k}\right|^2$$

$$= \frac{1}{8}\left|\delta_{\alpha\beta} + (1-2\kappa)(2\epsilon_{\alpha\beta} - \delta_{\alpha\beta})\right|\left|\frac{\partial x^r}{\partial a^k}\right|^2$$

$$= (1-2\kappa)^3\left|\epsilon_{\alpha\beta} + \frac{\kappa}{1-2\kappa}\,\delta_{\alpha\beta}\right|\left|\frac{\partial x^r}{\partial a^k}\right|^2$$

Therefore from equation 3.30, equation 3.31,

$$\kappa^3 + J_1\kappa^2 + J_2\kappa + J_3 = \left|\frac{\partial x^r}{\partial a^k}\right|^2\left[\kappa^3 + I_1\kappa^2(1-2\kappa) + I_2\kappa(1-2\kappa)^2 + I_3(1-2\kappa)^3\right].$$

Equating the coefficients of like powers of κ,

$$\left|\frac{\partial x^r}{\partial a^k}\right|^2 = 1/(1-2I_1+4I_2-8I_3), \tag{3.32}$$

$$J_1 = (I_1-4I_2+12I_3)/(1-2I_1+4I_2-8I_3),$$

$$J_2 = (I_2-6I_3)/(1-2I_1+4I_2-8I_3), \tag{3.33}$$

$$J_3 = I_3/(1-2I_1+4I_2-8I_3).$$

W is a function of J_1, J_2, J_3, and therefore of the η_{ij}, or by equation 3.33 it is a function of I_1, I_2, I_3, and therefore of the ϵ_{ij}. We may write

$$\delta W = \frac{\partial W}{\partial \epsilon_{\alpha\beta}}\,\delta\epsilon_{\alpha\beta} = \frac{\partial W}{\partial \eta_{\sigma\tau}}\,\delta\eta_{\sigma\tau} \tag{3.34}$$

The operator δ represents variation of a function of a given material point. It is therefore independent of the operators $\partial/\partial a^i$. Therefore

$$\delta\left(\frac{\partial x^i}{\partial a^j}\right) = \frac{\partial(\delta x^i)}{\partial a^j} = (\delta x^i)_{,r}\,\frac{\partial x^r}{\partial a^j}$$

Therefore by equation 3.28

$$\delta\eta_{\sigma\tau} = \tfrac{1}{2}\,\delta_{\alpha\beta}\left[(\delta x^{\beta})_{,r}\,\frac{\partial x^{\alpha}}{\partial a^{\sigma}}\,\frac{\partial x^{r}}{\partial a^{\tau}} + (\delta x^{\alpha})_{,r}\,\frac{\partial x^{r}}{\partial a^{\sigma}}\,\frac{\partial x^{\beta}}{\partial a^{\tau}}\right]$$

$$= \tfrac{1}{2}\left[(\delta x_{\alpha})_{,\beta} + (\delta x_{\beta})_{,\alpha}\right]\frac{\partial x^{\alpha}}{\partial a^{\sigma}}\,\frac{\partial x^{\beta}}{\partial a^{\tau}}$$

On the other hand, by equation 3.29,

$$\delta\eta_{\sigma\tau} = \delta\epsilon_{\alpha\beta}\,\frac{\partial x^{\alpha}}{\partial a^{\sigma}}\,\frac{\partial x^{\beta}}{\partial a^{\tau}} + \epsilon_{\alpha\beta}\left[(\delta x^{\beta})_{,r}\,\frac{\partial x^{\alpha}}{\partial a^{\sigma}}\,\frac{\partial x^{r}}{\partial a^{\tau}}\right.$$

$$\left. + (\delta x^{\alpha})_{,r}\,\frac{\partial x^{r}}{\partial a^{\sigma}}\,\frac{\partial x^{\beta}}{\partial a^{\tau}}\right]$$

$$= \left[\delta\epsilon_{\alpha\beta} + \epsilon_{\alpha r}(\delta x^{r})_{,\beta} + \epsilon_{r\beta}(\delta x^{r})_{,\alpha}\right]\frac{\partial x^{\alpha}}{\partial a^{\sigma}}\,\frac{\partial x^{\beta}}{\partial a^{\tau}},$$

so

$$\delta\epsilon_{\alpha\beta} = \tfrac{1}{2}\left[(\delta x_{\alpha})_{,\beta} + (\delta x_{\beta})_{,\alpha}\right] - \delta^{\sigma\tau}\,\epsilon_{\alpha\sigma}\,(\delta x_{\tau})_{,\beta}$$

$$- \delta^{\sigma\tau}\,\epsilon_{\sigma\beta}(\delta x_{\tau})_{,\alpha}$$

By the symmetry of the invariants,

$$\frac{\partial W}{\partial\epsilon_{\alpha\beta}} = \frac{\partial W}{\partial\epsilon_{\beta\alpha}}, \qquad \frac{\partial W}{\partial\eta_{\sigma\tau}} = \frac{\partial W}{\partial\eta_{\tau\sigma}},$$

so by equation 3.34

$$\delta W = \frac{\partial W}{\partial\eta_{\sigma\tau}}\,\frac{\partial x^{\alpha}}{\partial a^{\sigma}}\,\frac{\partial x^{\beta}}{\partial a^{\tau}}\,(\delta x_{\alpha})_{,\beta}$$

$$= \left[\frac{\partial W}{\partial\epsilon_{\alpha\beta}} - 2\delta^{\alpha\sigma}\,\epsilon_{\sigma\tau}\,\frac{\partial W}{\partial\epsilon_{\beta\tau}}\right](\delta x_{\alpha})_{,\beta}, \tag{3.35}$$

where W is a function of J_1, J_2, J_3 or of I_1, I_2, I_3.

$\delta W'$ is readily available from text books. It may be derived from [11; §19.41], and is given by

$$\delta W' = (M/\rho)(\delta^{\alpha\sigma}\dot{x}^{\beta}_{\cdot\sigma} + \delta^{\sigma\beta}\dot{x}^{\alpha}_{,\sigma})(\delta x_{\alpha})_{,\beta}, \tag{3.36}$$

where a dot above a symbol indicates differentiation with respect to time.

Finally we require an expression for $\delta W''$. Coulomb friction in the medium will be assumed to be a linear function of the expressions on the left-hand side of equation 3.18, which are measures of the distortion of the medium. In fact we will write

$$\delta W'' = (1/\rho) N^{\alpha\beta}(\delta x_\alpha)_{;\beta}, \tag{3.37}$$

where $N^{\alpha\beta}$ is a symmetric tensor. $N^{\alpha\beta} = 0$ whenever equation 3.12 holds. By the Second Law of Thermodynamics, $\delta \dot{W}'' \leq 0$, so by equation 3.37,

$$N^{\alpha\beta}\,\delta_{\beta\gamma}\,\dot{x}^{\gamma}_{,\alpha} \leq 0. \tag{3.38}$$

The form of $N^{\alpha\beta}$ will be considered more specifically in § 3.8.

Substituting equation 3.35-equation 3.37 into equation 3.23 and substituting the result into equation 3.22 we have, since $(\delta x_\alpha)_{,\beta}$ is arbitrary,

$$\tau^{ij} = T^{ij} + N^{ij} \tag{3.39}$$

where

$$\begin{aligned} T^{ij} &= \rho \frac{\partial W}{\partial \eta_{\sigma\tau}} \frac{\partial x^i}{\partial a^\sigma} \frac{\partial x^j}{\partial a^\tau} + M(\delta^{i\sigma}\dot{x}^j_{,\sigma} + \delta^{j\sigma}\dot{x}^i_{,\sigma}), \\ &= \rho\left(\frac{\partial W}{\partial \epsilon_{ij}} - 2\delta^{i\sigma}\epsilon_{\sigma\tau}\frac{\partial W}{\partial \epsilon_{j\tau}}\right) + M(\delta^{i\sigma}\dot{x}^j_{,\sigma} + \delta^{j\sigma}\dot{x}^i_{,\sigma}). \end{aligned} \tag{3.40}$$

3.7 THE TENSOR T^{ij}

W may be expanded in a triple Taylor series in I_1, I_2, I_3, only a few terms of which need usually be used. To terms of the third order we may write

$$\begin{aligned} \rho_0 W &= \gamma I_1 + \tfrac{1}{2}(\lambda + 2\mu + 3\gamma) I_1^2 - 2(\mu + \gamma) I_2 \\ &\quad + \tfrac{1}{6}(\lambda + \lambda' + \tfrac{7}{2}\gamma - 2h) I_1^3 + 2(h - \gamma) I_1 I_2 + q I_3, \end{aligned} \tag{3.41}$$

$\gamma, \lambda, \mu, \lambda', h, q$ being constants, and ρ_o being the density in the unstrained state.

By equation 3.30,

$$I_1 = \delta^{\alpha\beta}\epsilon_{\alpha\beta}, \quad I_2 = \tfrac{1}{2}I^2 - \tfrac{1}{2}\delta^{\alpha\sigma}\delta^{\beta\tau}\epsilon_{\alpha\tau}\epsilon_{\beta\sigma}$$

$$= \epsilon_{12}^2 + \epsilon_{23}^2 + \epsilon_{31}^2 - \epsilon_{11}\epsilon_{22} - \epsilon_{22}\epsilon_{33} - \epsilon_{33}\epsilon_{11} \tag{3.42}$$

$$I_3 = |\epsilon_{ij}|,$$

and

$$\frac{\partial I_1}{\partial \epsilon_{ij}} = \delta^{ij}, \quad \frac{\partial I_2}{\partial \epsilon_{ij}} = I_1\delta^{ij} - \delta^{i\alpha}\delta^{j\beta}\epsilon_{\alpha\beta},$$

$$\frac{\partial I_3}{\partial \epsilon_{ij}} = E^{ij}, \tag{3.43}$$

where E^{ij} is the cofactor of ϵ_{ij} in $|\epsilon_{ij}|$. From equation 3.41

$$\rho_o \frac{\partial W}{\partial \epsilon_{ij}} = \left[\gamma + (\lambda+\gamma)I_1 + (\lambda + \lambda' + \tfrac{3}{2}\gamma)I_1^2 + 2(h-\gamma)I_2\right]\delta^{ij}$$

$$+2\left[\mu+\gamma+(\gamma-h)I_1\right]\delta^{i\alpha}\delta^{j\beta}\epsilon_{\alpha\beta} + qE^{ij},$$

$$\rho_o\delta^{i\sigma}\epsilon_{\sigma\tau}\frac{\partial W}{\partial \epsilon_{j\tau}} = \left[\gamma + (\lambda+\gamma)I_1\right]\delta^{i\alpha}\delta^{j\beta}\epsilon_{\alpha\beta}$$

$$+2(\mu+\gamma)\delta^{i\sigma}\delta^{\alpha\tau}\delta^{j\beta}\epsilon_{\sigma\tau}\epsilon_{\alpha\beta},$$

dropping terms of the third order in the strain components.

The density ρ is that of the material in the strained state, and ρ_o is the density in the unstrained state. If dV and dV_o are volume elements in the two state,

$$\frac{\rho}{\rho_o} = \frac{dV_o}{dV} = \left|\frac{\partial a^k}{\partial x^\gamma}\right|,$$

Therefore by equation 3.32,

$$\rho = \rho_0(1-2I_1+4I_2-8I_3)^{1/2} , \tag{3.44}$$

so

$$\rho = \rho_0(1-I_1+2I_2-\tfrac{1}{2} I_1^2+\ldots).$$

By equation 3.40

$$\begin{aligned} T^{ij} = & \left[\gamma + \lambda I_1 + \lambda' I_1^2 + 2hI_2\right] \delta^{ij} \\ & +2\left[\mu-(h+\lambda+\mu)I_1\right] \delta^{i\alpha}\delta^{j\beta}\epsilon_{\alpha\beta} + qE^{ij} \\ & -4(\mu+\gamma)\delta^{i\alpha}\delta^{j\beta}\delta^{\sigma\tau}\epsilon_{\alpha\sigma}\epsilon_{\beta\tau} + N(\delta^{i\sigma}\dot{u}^j_{,\sigma} + \delta^{j\sigma}\dot{u}^i_{,\sigma}), \end{aligned} \tag{3.45}$$

where I_1 and I_2 are given by equation 3.42, where E^{ij} is the cofactor of ϵ_{ij} in $|\epsilon_{ij}|$, and where the Eulerian strain components are given in terms of the displacements

$$u^i = x^i - a^i \tag{3.46}$$

by

$$\epsilon_{ij} = \tfrac{1}{2}\delta_{i\sigma}u^\sigma_{,j} + \tfrac{1}{2}\delta_{j\sigma}u^\sigma_{,i} - \tfrac{1}{2}\delta_{\alpha\beta}u^\alpha_{,i}\, u^\beta_{,j} . \tag{3.47}$$

Equation 3.47 is an immediate consequence of equation 3.27.

3.8 THE TENSOR N^{ij}

The components N^{ij} vanish when equation 3.12 holds and when equation 3.38 cannot otherwise be satisfied. However in the yielding state they must be such that equation 3.13 holds. Since there are six components N^{ij}, it is clear that equation 3.13 cannot determine them entirely, and that more restrictions must be added.

At the present time adequate experimental information as to the proper choice of restrictions does not exist. On the other hand, how should one design experiments to obtain such information? Experimentation without theoretical guidance usually proves to have concerned itself with the wrong things. Usually it is not feasible to establish by direct experiment what the proper fundamental postulates of a physical theory should be. Its author must choose them on some basis which seems reasonable, and must develop

the theory to a point where experimental comparison becomes practicable. In this choice he may be guided not only by such incomplete experimental facts and physical principles as do apply, but also by the desirability of simplicity. The simplest theories should certainly be tried first.

A theory developed on this basis not only tells the experimenter what to look for, but often helps him form a new one if it fails.

Accordingly we shall use an argument based on the ideas of Coulomb friction to reduce the six unknown quantities N^{ij} to only one which may be determined by equation 3.13. From the standpoint of simplicity it is highly desirable that τ^{ij} and N^{ij} have the same principal directions.

At any given point in the medium construct the axes of principal stress and let y^1, y^2, y^3 denote the corresponding coordinates. Let τ_1, τ_2, τ_3 be the corresponding principal stresses ordered so that $\tau_1 \leq \tau_2 \leq \tau_3 \leq 0$. By §§3.2, 3.4, when the material yields and $\tau_1 < \tau_2 < \tau_3 < 0$, slipping may occur in the two planes whose normals have direction angles $\pi/2 \pm \kappa$, $\pi/2$, κ with respect to the y^1, y^2, y^3-axes, respectively. When $\tau_1 < \tau_2 = \tau_3 < 0$, slipping may occur in any of the infinite number of planes whose normals have the direction angle $\pi/2 - \kappa$ with respect to the y^1-axis. When $\tau_1 = \tau_2 < \tau_3 < 0$, slipping may occur in any of the infinite number of planes whose normals have the direction angle κ with respect to the y^3-axis. When $\tau_1 = \tau_2 = \tau_3$, slipping does not occur except in the singular case $\tau_1 = \tau_2 = \tau_3 = p = 0$. This condition is encountered on the boundary of a dry sand.

The energy loss due to Coulomb friction is proportional to the shearing stress in the plane of yield. We will assume an equi-partition of energy among the different possible planes of yield in each case. With each plane of yield we may associate a tensor defined in such a manner that in a rectangular coordinate system having the yield plane as one plane of reference, the only non-zero component is a shearing component in that plane, the direction of the shear being toward the principal axis of minimum stress. Giving all such shear components the same magnitude in the various possible planes of yield. N^{ij} will be defined as the sum of all such tensors.

The components of N^{ij} in the principal coordinate system of the tensor τ^{ij} may be expressed readily. N^{ij} has, in fact, the same principal axes, so we need only express the principal values N_1, N_2, N_3 of the tensor N^{ij}. It is a simple consequence of the tensor law of transformation that

$$N_1 = N, \quad N_2 = 0, \quad N_3 = -N \tag{3.48}$$

when $\tau_1 < \tau_2 < \tau_3 < 0$,

$$N_1 = N, \quad N_2 = -\tfrac{1}{2}N, \quad N_3 = -\tfrac{1}{2}N \tag{3.49}$$

when $\tau_1 < \tau_2 = \tau_3 < 0$,

$$N_1 = \tfrac{1}{2}N, \quad N_2 = \tfrac{1}{2}N, \quad N_3 = -N \tag{3.50}$$

when $\tau_1 = \tau_2 < \tau_3 < 0$, where N is a non-negative factor that vanishes whenever equation 3.12 holds or whenever equation 3.38 cannot otherwise be satisfied.

N is determined by means of condition equation 3.13. Let T_1, T_2, T_3 be the principal values of the tensor T^{ij}, i.e., the roots of the equation

$$\left| T^{ij} - T\,\delta^{ij} \right| = 0 \tag{3.51}$$

arranged in the order $T_1 \leq T_2 \leq T_3$. Equation 3.51 may also be written

$$T^3 - \Theta_1 T^2 + \Theta_2 T - \Theta_3 = 0, \tag{3.52}$$

where

$$\Theta_1 = T^{11} + T^{22} + T^{33}, \tag{3.53}$$

$$\Theta_2 = \begin{vmatrix} T^{11} & T^{12} \\ T^{12} & T^{22} \end{vmatrix} + \begin{vmatrix} T^{22} & T^{23} \\ T^{23} & T^{33} \end{vmatrix} + \begin{vmatrix} T^{33} & T^{31} \\ T^{31} & T^{11} \end{vmatrix}, \tag{3.54}$$

$$\otimes_3 = \begin{vmatrix} T^{11} & T^{12} & T^{13} \\ T^{12} & T^{22} & T^{23} \\ T^{13} & T^{23} & T^{33} \end{vmatrix} . \tag{3.55}$$

Then for $i = 1,2,3$,

$$\tau_i = T_i + N_i \tag{3.56}$$

In case $T_1 = T_2$ or $T_2 = T_3$, the effect of N in equation 3.49 or equation 3.50 is to bring τ_1 and τ_3 nearer together than are T_1 and T_3. If T_1 and T_3 are already near enough together to satisfy equation 3.12 with τ_1, τ_3 replaced by T_1, T_3, respectively, then $N = 0$. If not, N is determined by condition equation 3.13.

When $T_1 < T_2 < T_3$ the same process applies but with one complication. Consider the quantities $T_1 + N$, T_2, $T_3 - N$, which are of the forms of τ_1, τ_2, τ_3, respectively, when the regime of equation 3.48 applies. As N increases from zero, one of the two extreme quantities may come into coincidence with the middle quantity T_2 before N grows large enough to satisfy equation 3.13. In this case the regime of equation 3.48 will no longer apply, but either equation 3.49 or equation 3.50 must be used to bring τ_1 and τ_3 near enough together to satisfy equation 3.13.

It appears that a number of cases must be analysed separately, depending on the initial ranges of the quantities T_1, T_2, T_3. All may be verified by straightforward calculations to fit into the formula

$$\tau_i = T_i + M_i \delta \tag{3.57}$$

for $i = 1,2,3$, where

$$\begin{aligned} M_1 = \; & \frac{T_3 - T_1}{2} - K\left(p - \frac{T_1+T_3}{2}\right) \\ & + \frac{1+K}{3-K} \operatorname{pos}\left[T_2 - \frac{T_1+T_3}{2} - K\left(p - \frac{T_1+T_3}{2}\right)\right] \\ & - \frac{1+K}{3+K} \operatorname{pos}\left[\frac{T_1+T_3}{2} - T_2 - K\left(p - \frac{T_1+T_3}{2}\right)\right] , \end{aligned} \tag{3.58}$$

$$M_2 = -\frac{2}{3-K}\,\mathrm{pos}\left[T_2 - \frac{T_1+T_3}{2} - K\left(p - \frac{T_1+T_3}{2}\right)\right] + \frac{2}{3+K}\,\mathrm{pos}\left[\frac{T_1+T_3}{2} - T_2 - K\left(p - \frac{T_1+T_3}{2}\right)\right], \tag{3.59}$$

$$M_3 = -\frac{T_3-T_1}{2} + K\left(p - \frac{T_1+T_3}{2}\right) + \frac{1-K}{3-K}\,\mathrm{pos}\left[T_2 - \frac{T_1+T_3}{2} - K\left(p - \frac{T_1+T_3}{2}\right)\right] - \frac{1-K}{3+K}\,\mathrm{pos}\left[\frac{T_1+T_3}{2} - T_2 - K\left(p - \frac{T_1+T_3}{2}\right)\right], \tag{3.60}$$

where by equation 3.38,

$$\begin{aligned} \delta &= 0 \quad \text{if} \quad M_1\dot{u}^1_{,1} + M_2\dot{u}^2_{,2} + M_3\dot{u}^3_{,3} > 0, \\ \delta &= 1 \quad \text{if} \quad M_1\dot{u}^1_{,1} + M_2\dot{u}^2_{,2} + M_3\dot{u}^3_{,3} \leq 0. \end{aligned} \tag{3.61}$$

If y^1, y^2, y^3 are principal coordinates and x^1, x^2, x^3 are general coordinates, the components of the tensors τ^{ij} and T^{ij} are given by

$$\tau^{ij} = \tau_1 \frac{\partial x^i}{\partial y^1}\frac{\partial x^j}{\partial y^1} + \tau_2 \frac{\partial x^i}{\partial y^2}\frac{\partial x^j}{\partial y^2} + \tau_3 \frac{\partial x^i}{\partial y^3}\frac{\partial x^j}{\partial y^3}, \tag{3.62}$$

$$T^{ij} = T_1 \frac{\partial x^i}{\partial y^1}\frac{\partial x^j}{\partial y^1} + T_2 \frac{\partial x^i}{\partial y^2}\frac{\partial x^j}{\partial y^2} + T_3 \frac{\partial x^i}{\partial y^3}\frac{\partial x^j}{\partial y^3}. \tag{3.63}$$

3.9 BOUNDARY CONDITIONS

Boundary conditions are imposed at the exterior boundaries of the medium in the form of prescribed externally applied surface tractions or displacements. At the (in general moving) interior boundaries between the conservative and yielding regimes, the displacement components u^i must be continuous functions of position and time.

In the particular case when the exterior boundary is free, acted upon by no externally applied surface tractions,

the compressive traction normal to the boundary is p, the surface-tension pressure. Suppose x^1 is a rectangular coordinate normal to the surface. Then by equation 3.62,

$$(\tau_1 - p)\left(\frac{\partial x^1}{\partial y^1}\right)^2 + (\tau_2 - p)\left(\frac{\partial x^1}{\partial y^2}\right)^2 + (\tau_3 - p)\left(\frac{\partial x^1}{\partial y^3}\right)^2$$
$$= \tau'' - p = 0. \tag{3.64}$$

The medium cannot support tension greater than p, so

$$\tau_1 - p \leq \tau_2 - p \leq \tau_3 - p \leq 0. \tag{3.65}$$

The quantities $\partial x^1/\partial y^1$, $\partial x^1/\partial y^2$, $\partial x^1/\partial y^3$ cannot vanish simultaneously so by equation 3.64 one, at least, of the quantities $\tau_1 - p$, $\tau_2 - p$, $\tau_3 - p$ must vanish. In particular, by equation 3.65, $\tau_3 - p = 0$. Therefore $\tau_1 - p = 0$, for neither equation 3.12 nor equation 3.13 could otherwise be satisfied. Then by equation 3.65, $\tau_2 - p = 0$.

Theorem 3.1 The stress distribution on a free exterior boundary is completely characterized by

$$\tau_1 = \tau_2 = \tau_3 = p \tag{3.66}$$

or by

$$\tau^{ij} = pg^{ij}. \tag{3.67}$$

Equation 3.67 is derived from equation 3.66 and equation 3.62.

Theorem 3.2 Whenever equation 3.67 holds, then

$$T^{11} + T^{22} + T^{33} = 3p. \tag{3.68}$$

Conversely, when equation 3.68 holds, then

$$\tau^{ij} = T^{ij} \text{ if } (p-T_1)\dot{u}^1_{,1} + (p-T_2)\dot{u}^2_{,2} + (p-T_3)\dot{u}^3_{,3} > 0, \tag{3.69}$$

$$\tau^{ij} = pg^{ij} \text{ if } (p-T_1)\dot{u}^1_{,1} + (p-T_2)\dot{u}^2_{,2} + (p-T_3)\dot{u}^3_{,3} \leq 0. \tag{3.70}$$

Proof. By equation 3.63, $T^{11} + T^{22} + T^{33} = T_1 + T_2 + T_3$.

To derive equation 3.68 from equation 3.67, note that equation 3.67 is equivalent to equation 3.66. Equation 3.68 then follows by adding equations 3.58, 3.59 and 3.60 and substituting into equation 3.57.

Conversely, suppose equation 3.68 holds. Then

$$T_1 + T_2 + T_3 = 3p,$$

and

$$T_2 - \frac{T_1 + T_3}{2} - K\left(p - \frac{T_1+T_3}{2}\right) = (3-K)\left(p - \frac{T_1+T_3}{2}\right),$$

$$\frac{T_1+T_3}{2} - T_2 - K\left(p - \frac{T_1+T_3}{2}\right) = -(3+K)\left(p - \frac{T_1+T_3}{2}\right).$$

Substituting these into equations 3.58-3.60 and noting that pos(x) - pos(-x) = x, we easily see that $M_1 = -T_1+p$. Equations 3.69 and 3.70 then follow from equations 3.57 and 3.61.

In the yielding regime the condition in equation 3.70 is satisfied. It then follows that equation 3.68 is equivalent to equation 3.67, so that the boundary conditions on a free exterior surface in the yielding regime are completely characterized by condition equation 3.68.

3.10 UNI-DIRECTIONAL DISPLACEMENTS IN DRY SOIL

Possibly the simplest case to analyse is that in which all displacements are in only one direction, say parallel to the x-axis in an x,y,z-rectangular coordinate system, and depend on x and t (time) only. This case arises, for example, when a rigid cylindrical container is filled with soil and compressed by a piston at one end.

We take $x^1 = x$, $x^2 = y$, $x^3 = z$. By equation 3.47,

$$\epsilon_{11} \equiv \epsilon = u_x - \tfrac{1}{2} u_x^{\,2}, \qquad (3.71)$$

and all components of strain are zero. By equation 3.42, $I_1 = \epsilon$, $I_2 = I_3 = 0$. Also $E^{ij} = 0$. By equation 3.45, since $M = 0$,

$$
\begin{aligned}
T^{xx} &= \gamma + (\lambda + 2\mu)\,\epsilon + (\lambda' + 2\mu')\,\epsilon^2 , \\
T^{yy} = T^{zz} &= \gamma + \lambda\epsilon + \lambda'\epsilon^2 , \qquad (3.72) \\
T^{xy} = T^{yz} &= T^{zx} = 0,
\end{aligned}
$$

where $\mu' = -(h + \lambda + 3\mu + 2\gamma)$.

We are interested in compression only, so $\epsilon < 0$. Then for $\mu' \geqslant 0$ or for ϵ sufficiently small $T^{xx} < T^{yy} = T^{zz}$. Therefore by equations 3.52-3.55, equation 3.72, $T_1 = T^{xx}$, $T_2 = T_3 = T^{yy} = T^{zz}$. Also, of course, $\tau^{xx} = \tau_1$, $\tau^{yy} = \tau^{zz} = \tau_2$, $\tau^{xy} = \tau^{yz} = \tau^{zx} = 0$. By equations 3.57-3.61, equation 3.71, equation 3.72, since $p = 0$,

$$
\begin{aligned}
\tau^{xx} &= \frac{1+K}{1-K}\left[3\gamma + (3\lambda + 2\mu)\,\epsilon + (3\lambda' + 2\mu')\,\epsilon^2\right] , \\
\tau^{yy} = \tau^{zz} &= \frac{1-K}{3-K}\left[3\gamma + (3\lambda + 2\mu)\,\epsilon + (3\lambda' + 2\mu')\,\epsilon^2\right]
\end{aligned}
\qquad (3.73)
$$

if

$$
K\gamma + \left[\mu - K(\lambda + \mu)\right](-\epsilon) + \left[-\mu' + K(\lambda' + \mu')\right]\epsilon^2 \geqslant 0 \quad \text{and} \quad \dot{\epsilon} < 0, \qquad (3.74)
$$

while

$$
\begin{aligned}
\tau^{xx} &= \gamma + (\lambda + 2\mu)\,\epsilon + (\lambda' + 2\mu')\,\epsilon^2, \\
\tau^{yy} = \tau^{zz} &= \gamma + \lambda\epsilon + \lambda'\epsilon^2
\end{aligned}
\qquad (3.75)
$$

otherwise.

Equations 3.73, 3.74 correspond to the yielding regime, and equation 3.75 to the elastic regime. It will be convenient to discuss the plane in which $-\tau^{xx}$ is plotted as ordinate against $-\epsilon$ as abscissa. Only the first quadrant will be of interest.

In successive transitions from the elastic to the yielding condition and back it will usually be necessary to change the constants appearing in equations 3.73, 3.75 in successive appearances of the same regime. It is clear

that at the beginning of the initial deformation of a soil, $\tau^{xx} = 0$ when $\epsilon = 0$, so $\gamma = 0$. On the other hand, for later deformations when the soil has received permanent set we must have $\gamma \neq 0$. Since the adjustment of γ is necessary in any case, since this is by far the simplest constant to adjust, and in the absence of experimental evidence to the contrary, we will henceforth in the present paper adjust only this constant, regarding it therefore as a parameter and considering $\lambda, \mu, \lambda', \mu'$ to be constants of the material which are not changed in spite of the compaction of the material. Of course this hypothesis may require modification when more is known about soil mechanics, but for the present its simplicity recommends it.

In the $(-\tau^{xx}, -\epsilon)$ plane, the first of equation 3.73 represents a family of yielding curves obtained by varying the parameter γ, and the first of equation 3.75 similarly represents a family of elastic curves. The second condition in equation 3.74 can only be satisfied if the point $(-\tau^{xx}, -\epsilon)$ is moving to the right in this plane, so yielding can only occur when this is the case. A reversal of direction initiates an elastic regime, the point then following the particular member of the elastic family passing through the point where the reversal occurred.

However for yielding to occur it is not only necessary that the point $(-\tau^{xx}, -\epsilon)$ be moving to the right, but also that the first condition in equation 3.74 hold. We will now determine the part of the $-\tau^{xx}, -\epsilon$ plane where this condition holds.

Consider the point $(-\tau^{xx}, -\epsilon)$. If this is on a yield curve, the corresponding value of γ is obtained by solving the first of equation 3.73. If this is substituted into the first of equation 3.74 we get

$$-\tau^{xx} \leq (1+1/K)\left[\mu(-\epsilon) - \mu'(-\epsilon)^2\right]. \qquad (3.76)$$

It follows that yielding, described by the regime of equation 3.73, can occur only between the parabola

$$-\tau^{xx} = (1+1/K)\left[\mu(-\epsilon) - \mu'(-\epsilon)^2\right] \qquad (3.77)$$

and the $-\epsilon$ axis, and only for points $(-\tau^{xx}, -\epsilon)$ moving to the right. All points moving to the left, and all points in the first quadrant outside this parabola correspond to elastic deformation, described by the regime of equation 3.75

The case of practical interest is that in which

$$K \leq \mu/(\lambda+\mu) \quad \text{or} \quad k \leq \mu/\sqrt{\left[\lambda(\lambda+2\mu)\right]}\ , \tag{3.78}$$

in which, as will be seen later, the seismic velocity in the yielding state is lower than that in the elastic state. In this case the elastic curves of equation 3.75 have, at $-\epsilon = 0$, a greater slope than do the yielding curves. A typical situation is shown in Fig. 3.1. The dotted curve is the yield limit parabola of equation 3.77. The solid lines are members of the elastic family of equation 3.75, and the dashed lines are members of the yielding family of equation 3.73.

Now suppose the end of the dirt column is rammed a number of times, each time $-\tau^{xx}$ being raised to a value T and then returning t zero. The stress-strain curve followed by the phenomenon zig-zags up and down in Fig. 3.2, tending toward the right-hand side of the yield limit parabola, and approaching the elastic line through $P(0,\mu/\mu')$, where the yield limit parabola meets the $-\epsilon$ axis. This model therefore exhibits the familiar behavior of soil when tamped to a more solid condition.

3.11 A ONE-DIMENSIONAL WAVE PROBLEM

We may apply the theory of §3.10 to study the one-dimensional transmission through a semi-infinite dirt column of a wave due to the application at the end of the column of a force T per unit area for an interval of time t_0, the force being then removed. For simplicity we will employ the linear theory obtained by setting $\lambda' = 0$, $\mu' = 0$ in §3.10 and $\rho = \rho_0$ in equation 3.20. This implies that the entire stress-strain history of the material remains near the lower left-hand corner of the parabolic yielding region in Fig. 3.1.

The external forces are inertial: $F^x = -u_{tt}$, $F^y = 0$, $F^z = 0$ in the coordinate system of §3.10. The origin is taken at the end of the column. By equation 3.71 we may approximate

$$\epsilon = u_x \tag{3.79}$$

By equations 3.20, 3.73, 3.75,

$$c^2u_{xx} - u_{tt} = 0 \quad \text{when } u_{xt} > 0, \tag{3.80}$$

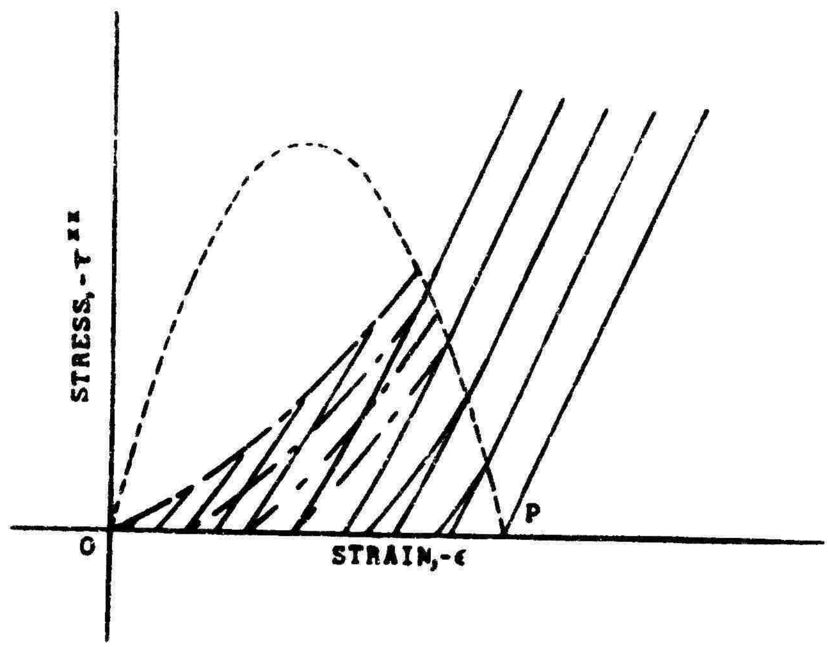

FIG. 3.1 Yield Limit Parabola

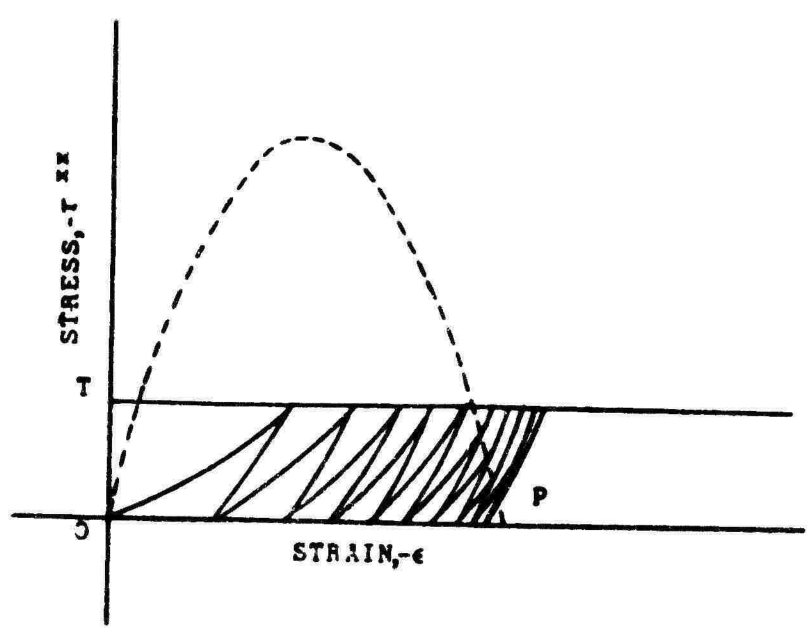

FIG. 3.2 Soil Tamping

and

$$c'^2 u_{xx} - u_{tt} = 0 \quad \text{when} \quad u_{xt} < 0, \tag{3.81}$$

where

$$c^2 = (\lambda + 2\mu)/\rho_o, \quad c'^2 = (3\lambda + 2\mu)(1+K)/\left[(3-K)\rho_o\right] \tag{3.82}$$

Instead of the stress-strain diagram of Fig. 3.1, we now have one like that shown in Fig. 3.3, where the straight lines (of slope c'^2) parallel to OP are yielding curves, and the straight lines (of slope c^2) parallel to PR are elastic.

In this problem the initial compression of the medium must correspond to a displacement along the line OP in Fig. 3.3, carrying, say, to the point P. The following decompression will be along the line PR. During the compression the yielding regime described by equations 3.73, 3.79, 3.81 holds, while during the decompression the elastic regime described by equations 3.75, 3.79, 3.80 holds.

Writing $u = u(x,t)$ we may now give boundary conditions at the end $x = 0$ of the column. During the interval $0 < t < t_o$, $-\tau^{xx} = T$, corresponding to the point P, say, in Fig. 3.3. The corresponding strain is given by equation 3.73. For $t > t_o$, $-\tau^{xx} = 0$, corresponding to the point R in Fig. 3.3. The corresponding strain is given by equation 3.75. In view of equation 3.79 we have

$$u_x(0,t) = -\frac{T}{3\lambda + 2\mu}\,\frac{3-K}{1+K} \quad \text{when } 0 < t < t_o,$$

$$u_x(0,t) = \frac{T}{\lambda + 2\mu} - \frac{T}{3\lambda + 2\mu}\,\frac{3-K}{1+K} \quad \text{when } t > t_o. \tag{3.83}$$

In the yielding regime, the wave equation 3.81 holds, so the initial disturbance is propagated in the positive x-direction with a velocity c'. In fact, $u = \text{fun}(x - c't)$. By the first of equation 3.83,

$$u(x,t) = +\frac{T}{3\lambda + 2\mu}\,\frac{3-K}{1+K}\,\text{pos}(c't - x) \tag{3.84}$$

in the yielding regime. The yielding regime can be taken

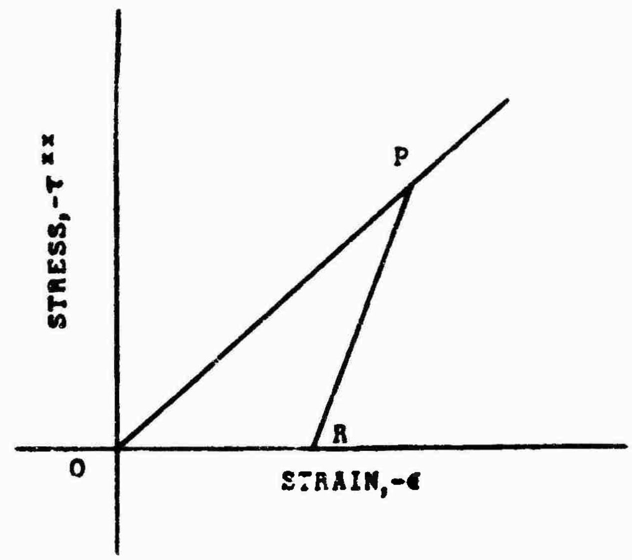

FIG. 3.3 Stress-Strain Diagram

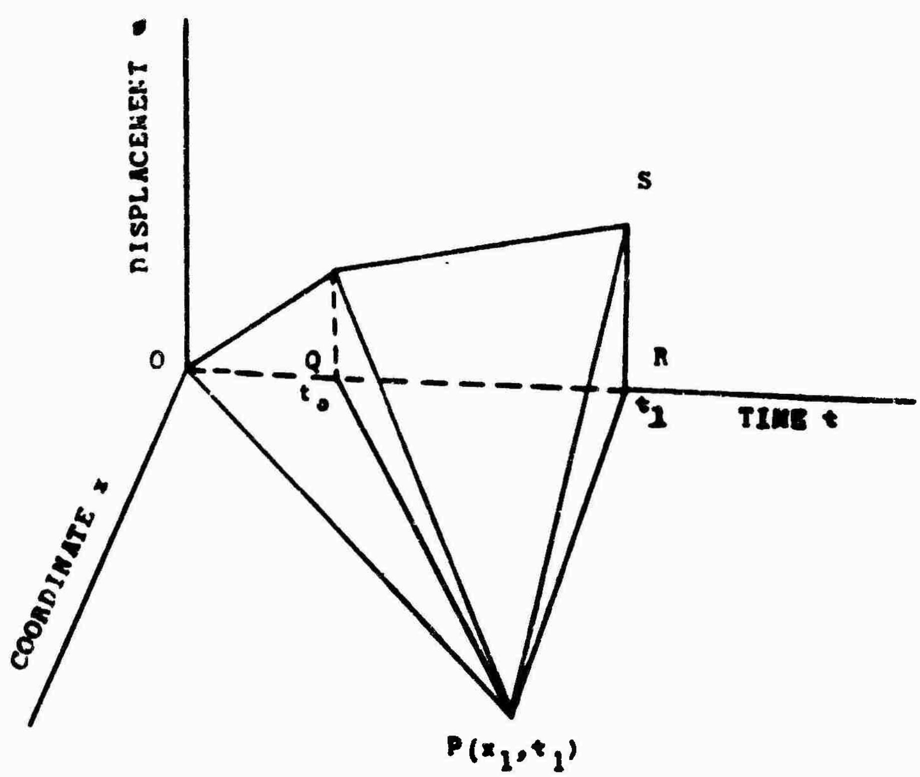

FIG 3.4 Displacement Diagram

as holding until the decompressional wave arrives. This begins at time $t = t_o$ at $x = x_o$ and propagates in the positive x-direction with velocity c. It overtakes the compressional wave at time $t = t_o c/(c-c')$ at $x = t_o cc'/(c-c')$.

Since equation 3.84 is linear in x and t, it suggest that the elastic regime might be described in terms of another linear function. This, in fact, proves to be the case. The constants are determined by the second of equation 3.83 and the fact that u(x,t) must be continuous at $x = c(t-t_o)$. We have altogether,

$$u(x,t) = \frac{T}{3\lambda + 2\mu} \frac{3-K}{1+K} \ (c't-x) \qquad (3.85)$$

when $0 < t < t_o c/(c-c')$ and pos $\left[c(t-t_o)\right] < x < c't$, and

$$u(x,t) = \frac{T}{3\lambda + 2\mu} \frac{3-K}{1+K} \left[\left(1-\frac{c'}{c}\right)c't - \left(1-\frac{c'^2}{c^2}\right)x + \frac{c'^2}{c} t_o\right] \qquad (3.86)$$

when $t_o < t < t_o c/(c-c')$ and $0 < x < c(t-t_o)$.

In equation 3.85 and equation 3.86 u is plotted as a function of x and t in Fig. 3.4. Here

$$t_1 = t_o c/(c-c'), \quad x_1 = c't_1. \qquad (3.87)$$

For $t > t_1$ the material is at rest, maintaining the displacement profile PRS it had at $t = t_1$. This profile then represents the permanent displacement of the soil due to the original impulse. For $0 < t < t_1$, $x > c't$, $u = 0$ in Fig. 3.4.

The energy imparted to the soil by the impressed force has spent itself in the time t_1. The deformation of the soil due to this excitation does not penetrate beyond $x = x_1$. A permanent deformation of the soil is left which is a linear function of the distance from the source.

3.12 A GRAPHICAL SOLUTION OF A ONE DIMENSIONAL

Suppose the problem of § 3.11 is extended in the sense that at the end x = 0 of the column a variable force T(t) per unit area is applied. By plotting $-\tau^{xx}$ in three dimensions against x and t, a graphical representation of the stress may be made. The curve $-\tau^{xx} = T(t)$ is plotted in the $(-\tau^{xx}, t)$ plane as in Fig. 3.5. Where this curve has

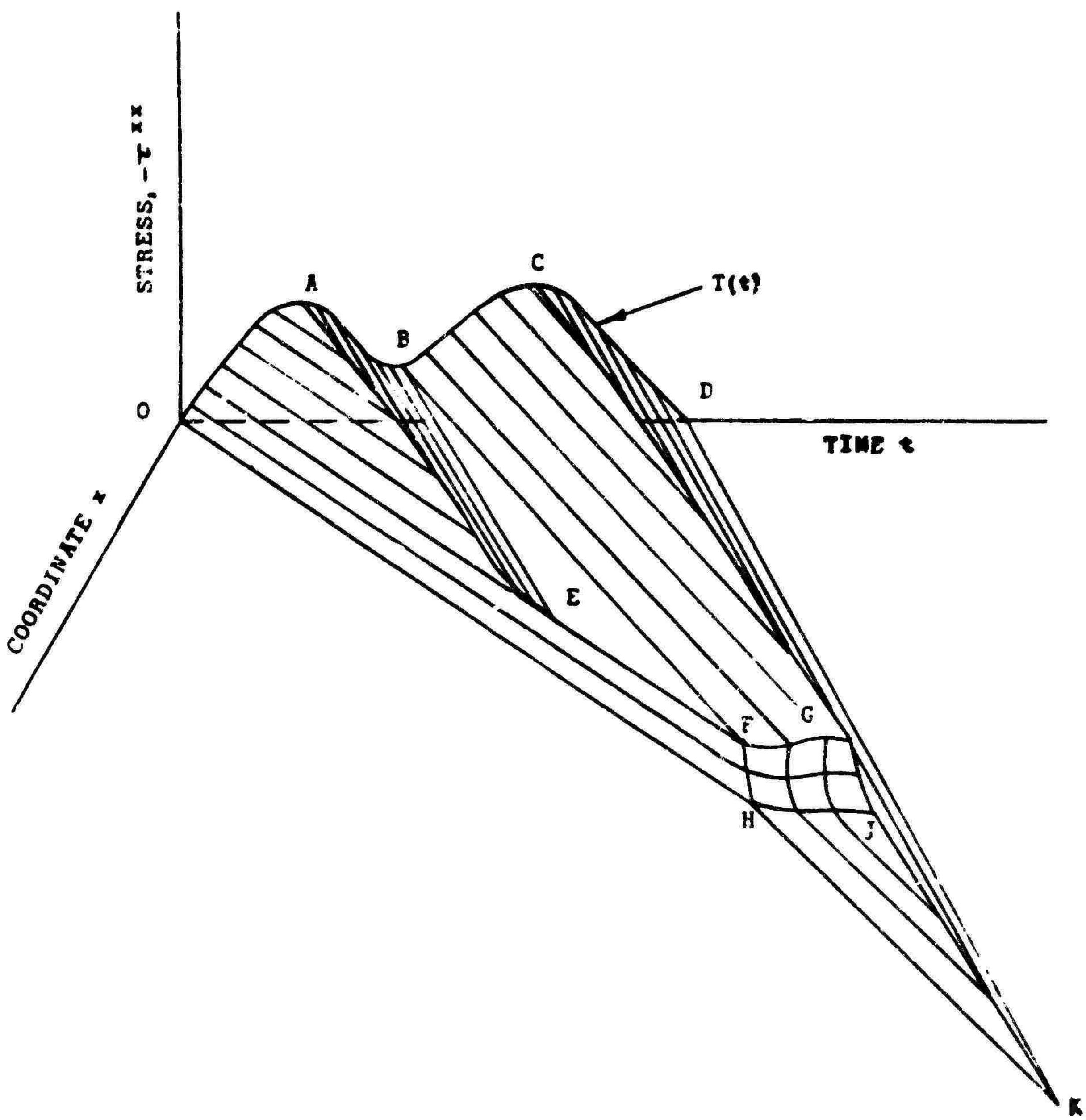

FIG. 3.5 Stress Diagram

a positive slope, the stress is increasing and the yielding regime applies inside the yield limit parabola. When this curve has a negative slope the elastic regime applies.

When the maxima of T(t) are fairly low the segments of the stress-strain curve corresponding to $-\tau^{xx} = T(t)$ may be approximated by straight lines with appropriately chosen average slopes. The slopes of successive elastic or plastic stress-strain curves in this diagram may differ in different parts of the plastic region.

Impulses will be propagated through the medium with the velocities indicated by the square roots of the slopes in Fig. 3.6. Let A,B,C be maximum, minimum, maximum, respectively on the T(t) curve in Fig. 3.5. Impulses during the compression regime OA on T(t) will propagate with a velocity equal to the square root of the slope OA in Fig. 3.6, and will generate a ruled surface OAE in Fig. 3.5. Impulses during the decompression regime AB propagate with the higher velocity equal to the square root of the slope of AB in Fig. 3.6. These also generate a ruled surface ABE in Fig. 3.5, which intersects the first ruled surface OAE in the line AE. Impulses from the compression regime BC and from the decompression regime CD in Fig. 3.5 propagate with velocities similarly obtained from BC and CD in Fig. 3.6, and generate similar ruled surfaces BCFG or HJK and CDK, which intersect along a line CGK. In this second case, however, there is a difference. The particular curve T(t) chosen for the diagram was such that part of the original compression was unneutralized by the succeeding decompression. This reinforced the second compression, resulting in the superposition region FGJH which is not a ruled surface.

The horizontal surface BLF is interesting and is a result of the discontinuity in slope between AB and BC in Fig. 3.6.

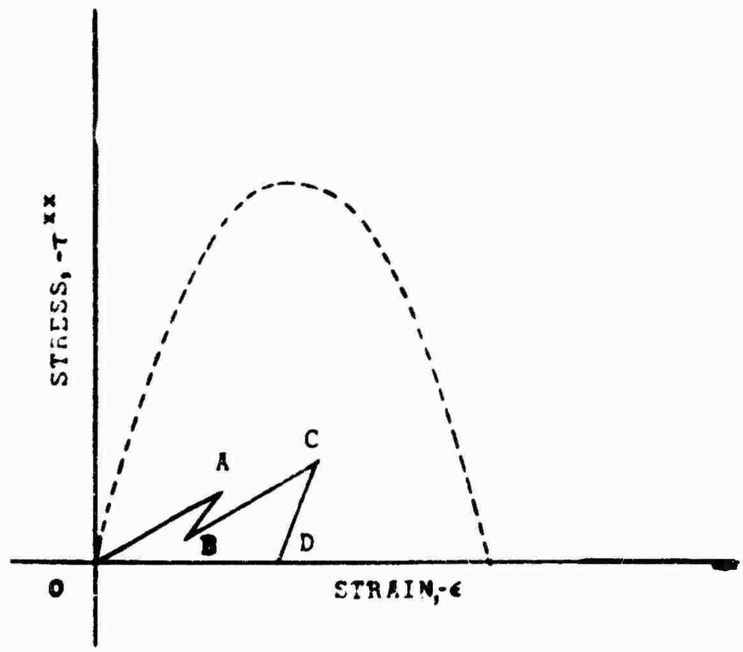

FIG 3.6 Stress-Strain Curve

CHAPTER 4

AN OPERATIONAL SOLUTION FOR DISPLACEMENTS IN AN ELASTIC HALF-SPACE

4.1 INTRODUCTION

In this chapter we shall develop general solutions for the vibratory motions of an elastic half-space with given boundary conditions in the form of complex inversion integrals of the Laplace transform.

The solution will be developed in such a manner that any type of pressure distribution may be given on the surface of a small sphere below the boundary surface of the elastic half-space. In the past, many considerations have been made for a continuous harmonic point source in the half-space wherein final study is made of the complicated integrals that arise in order to determine the various types of waves propagated by reflection back into the half-space as well as along the surface. In order that these procedures be of any value in the study of explosive disturbances in such a medium, the solutions due to a harmonic source would have to be formed into a pulse by means of Fourier procedures. Due to the complicated nature of the solution in the form of definite integrals, this is seldom done.

Using the solutions developed in this chapter, one can consider the effects in the half-space and in the surface layer of a unit pulse function applied to the source below the surface, or any other pressure-time function as applied to the interior surface of the spherical cavity.

Professor E. Pinney, at the University of California, has considered the same problem from another point of view. His paper is soon to be published with a considerable amount of numerical work which has been done by a computing project.

One should point out that in order to solve the boundary value problem exactly where the spherical source below

the boundary surface has a finite radius, one must consider the tertiary reflections from the spherical source. However, we shall assume that if the spherical source is sufficiently small we may neglect the tertiary reflections as being less dominant features on the instrument records compared to the effect of the primary and secondary waves when measurements are being considered fairly close in to the source.

Specifically, we consider the exact solution for any pressure distribution on a small spherical surface in an infinite medium and show how generally this may be used to solve exactly the problem in the half-space.

It is well known that if an elastic body is suddenly loaded the body takes up a mode of vibration about the position of static displacement which the body would assume if the load had been applied slowly. Therefore, if we have chosen a unit pulse in the elastic half-space, we know the displacements in the medium oscillate about the position of static equilibrium. Therefore, we could solve the static part of the problem and know this part of the displacement beforehand; however, the general solution presented will automatically include this term in the integral solution.

4.2 STRESS FUNCTIONS FOR THE ELASTIC EQUATIONS OF MOTION IN COMPLEX LAPLACE TRANSFORM INTEGRAL FORM.

If u, v and w are the rectangular components of displacement in the x, y, and z directions respectively, then the equations of motion are [9]

$$\rho \frac{\partial^2}{\partial t^2}(u,v,w) = (\lambda+\mu)\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\Delta\right]+\mu\nabla^2(u,v,w) \tag{4.1}$$

where ρ = density,

$$\Delta = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} \tag{4.2}$$

is the so-called dilation, and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

the Laplacian operator. Let

$$u = \left(\frac{\partial \Phi}{\partial x} + u'\right) T(t)$$

$$v = \left(\frac{\partial \Phi}{\partial y} + v'\right) T(t) \tag{4.3}$$

$$w = \left(\frac{\partial \Phi}{\partial z} + w'\right) T(t)$$

where $\Phi = \Phi(x,y,z)$ is a "stress function". Then by equation 4.2 we find

$$\Delta = \left(\nabla^2 \Phi + \Delta'\right) T(t) \tag{4.4}$$

where

$$\Delta' = \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \tag{4.5}$$

For each component we have typically

$$\rho \frac{\partial^2}{\partial t^2} (\Phi_x + u')T(t) = (\lambda + \mu) \nabla^2 \Phi_x + \frac{\partial}{\partial x}(\Delta') T(t) + \mu \nabla^2 \Phi_x \cdot T(t) + \mu \nabla^2 u' \cdot T(t)$$

and if we set $\Delta' = 0$, then we find

$$\rho \frac{T''}{T} = \frac{(\lambda + 2\mu) \nabla^2 \Phi_x + \mu \nabla^2 u'}{\varphi_x + u'} = -m^2 \tag{4.6}$$

where m is some arbitrary constant. Separating, we have

$$\rho T'' + m^2 T = 0, \tag{4.7}$$

m real or complex, and

$$\left\{ \begin{array}{lll} (\nabla^2 + \alpha^2) & \Phi & \text{(a)} \\ (\nabla^2 + \beta^2) & u' = 0 & \text{(b)} \end{array} \right. \tag{4.8}$$

where

$$\alpha^2 = \frac{m^2}{\lambda+2\mu} \quad \text{and} \quad \beta^2 = \frac{m^2}{\mu} \tag{4.9}$$

A particular solution of $\Delta' = 0$, suitable for the symmetry desired, is

$$u' = \frac{\partial^2 \Psi}{\partial x \partial z} \quad , \quad v' = \frac{\partial^2 \Psi}{\partial y \partial z} \quad , \quad w' = \frac{\partial^2 \Psi}{\partial z^2} + \beta^2 \Psi \tag{4.10}$$

where $\Psi = \Psi(x,y,z)$ is a "stress function" and equation 4.8 requires that

$$\left(\Delta^2 + \beta^2\right) \Psi = 0 \tag{4.11}$$

Finally, we may express each component of displacement as

$$\begin{aligned} u &= (\Phi_x + \Psi_{xz})T(t) \\ v &= (\Phi_y + \Psi_{yz})T(t) \\ w &= (\Phi_z + \Psi_{zz} + \beta^2 \Psi)T(t) \end{aligned} \tag{4.12}$$

where $T(t)$ satisfies equation 4.7, $\Phi(x,y,z)$ and $\Psi(x,y,z)$ satisfy equation 4.8 (a) and 4.11 respectively, with α and β defined by equation 4.9. When we build solutions which satisfy prescribed boundary conditions, we must require that $u,v,w \rightarrow 0$ for $z \rightarrow \infty$. If we choose the functional form for $T(t)$ as

$$T(t) = e^{st} \tag{4.13}$$

then the solutions 4.12 can be written in the complex inversion transform as

$$\left\{ \begin{aligned} u &= \frac{1}{2\pi i} \lim_{\beta \to 0} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{st} \left[\Phi_x + \Psi_{xz} \right] ds = L^{-1}(\Phi_x + \Psi_{xz}) \\ v &= \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{st} \left[\Phi_y + \Psi_{yz} \right] ds = L^{-1}(\Phi_y + \Psi_{yz}) \\ w &= \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{st} \left[\Phi_z + \Psi_{zz} - \frac{s^2}{v_s^2} \Psi \right] ds = L^{-1}(\Phi_z + \Psi_{zz} - \frac{s^2}{v_s^2} \Psi) \end{aligned} \right. \tag{4.14}$$

where $\Phi(x,y,z,s)$ and $\Psi(x,y,z,s)$ are continuous differentiable functions of s, analytic in the half-plane $Re(s) \geqslant \gamma$ expressed in terms of its values along the line $\gamma - i\beta \to \gamma + i\beta$ for some suitable fixed γ. The functions Φ and Ψ satisfy

$$\left(\nabla^2 - s^2/v_c^2\right)\Phi = 0 \qquad \text{and} \qquad \left(\nabla^2 - s^2/v_c^2\right)\Psi = 0 \tag{4.15}$$

where we have introduced

$$v_c = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad v_s = \sqrt{\frac{\mu}{\rho}} \tag{4.16}$$

v_c = velocity of the compressional wave and v_s = velocity of the shear wave. Incidentally, since

$$\lambda = \rho\left(v_c^2 - 2v_c^2\right)$$

and $\rho, \lambda > 0$, we conclude

$$\frac{v_c}{v_s} > \sqrt{2}\,. \tag{4.17}$$

In particular, we find that

$$\Delta = \frac{1}{2\pi i} \lim_{\beta \to \infty} \cdot \frac{1}{v_c^2} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{st}\, s^2\, \Phi\, ds = \frac{L^{-1}}{v_c^2}(s^2\,\Phi) \tag{4.18}$$

Since the stresses are

$$\begin{cases} \tau_{xx} = \lambda\Delta + 2\mu\,\frac{\partial u}{\partial x} \\ \tau_{yy} = \lambda\Delta + 2\mu\,\frac{\partial v}{\partial y} \\ \tau_{zz} = \lambda\Delta + 2\mu\,\frac{\partial w}{\partial z} \end{cases} \qquad \begin{cases} \tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \end{cases} \tag{4.19}$$

we calculate and find

$$\begin{cases} \tau_{xx} = L^{-1}\left\{\left[\frac{\lambda s^2}{v_c^2}\Phi + 2\mu\frac{\partial^2}{\partial x^2}\left(\Phi + \Psi_z\right)\right]\right\} \\ \tau_{yy} = L^{-1}\left\{\left[\frac{\lambda s^2}{v_c^2}\Phi + 2\mu\frac{\partial^2}{\partial y^2}\left(\Phi + \Psi_z\right)\right]\right\} \\ \tau_{zz} = L^{-1}\left\{\left[\frac{\lambda s^2}{v_c^2}\Phi + 2\mu\frac{\partial^2}{\partial z^2}\left(\Phi + \Psi_z\right) - \frac{2\mu s^2}{v_c^2}\Psi_z\right]\right\} \end{cases} \tag{4.20}$$

$$\begin{cases} \tau_{xy} = 2\mu L^{-1}\left\{\frac{\partial^2}{\partial x\partial y}\left(\Phi + \Psi_z\right)\right\} \\ \tau_{yz} = \mu L^{-1}\left\{\left[2\frac{\partial^2}{\partial y\partial z}\left(\Phi + \Psi_z\right) - \frac{s^2}{v_s^2}\Psi_y\right]\right\} \\ \tau_{xz} = \mu L^{-1}\left\{\left[2\frac{\partial^2}{\partial x\partial z}\left(\Phi + \Psi_z\right) - \frac{s^2}{v_s^2}\Psi_x\right]\right\} \end{cases} \tag{4.21}$$

Then for some type of boundary conditions which have stresses ~~that~~ are prescribed functions of time, we write

$$\tau_{ii} = L^{-1}\left\{f(s)\right\} - \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} e^{st} f(s)\, ds\,. \tag{4.22}$$

The function f(s) will therefore be determined by the boundary conditions on some portion of the elastic medium. To complete the solution we take linear combinations of the fundamental solutions of equation 4.15. The coefficients of these linear combinations are determined from the boundary conditions using equations 4.20 and 4.21. These coefficients are replaced in equations 4.20 and 4.21 and we have the complete solution. Let us consider in some detail the case in spherical coordinates where the essential coordinate is radial.

4.3 RADIAL MOTION

In this section we shall develop the solution for the case of an arbitrary radial pressure on the interior of a small spherical cavity in an infinite elastic medium. In particular we shall derive the solution by the method of section 4.2 for a unit pulse in the spherical cavity. We have basically,

$$\Delta = \frac{\partial U}{\partial R} + \frac{2U}{R} \tag{4.23}$$

with U(R) = radial displacement,

$$e_{RR} = \frac{\partial U}{\partial R} \quad , \quad e_{\theta\theta} = \frac{U}{R} \quad ,$$

the strains and the stresses

$$\begin{cases} T_{RR} = (\lambda + 2\mu) \dfrac{\partial U}{\partial R} + 2\lambda \dfrac{U}{R} \\ \\ T_{\theta\theta} = \lambda \dfrac{\partial U}{\partial R} + 2(\lambda + \mu) \dfrac{U}{R} \quad . \end{cases} \tag{4.24}$$

The equation of motion is

$$\rho \frac{\partial^2 U}{\partial t^2} = (\lambda + 2\mu) \frac{\partial}{\partial R} (\Delta) \tag{4.25}$$

and if we set

$$U = \frac{\partial \otimes}{\partial R} \quad , \quad \Delta = \frac{1}{R} \frac{\partial^2}{\partial R^2} (R \cdot \otimes) \quad , \tag{4.26}$$

then $R \cdot \otimes$ satisfies

$$\rho \frac{\partial^2}{\partial t^2} (R \otimes) = (\lambda + 2\mu) \frac{\partial^2}{\partial R^2} (R \otimes) . \tag{4.27}$$

Let us set

$$R \cdot \otimes = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{st} Q(R) \, ds = L^{-1} \left[Q(R) \right] \tag{4.28}$$

Then equation 4.27 gives

$$L^{-1} \left[s^2 Q - v_c^2 \frac{d^2 Q}{dR^2} \right] = 0 \quad ,$$

or Q(R) satisfies

$$\frac{d^2 Q}{dR^2} - \frac{s^2}{v_c^2} Q = 0 \quad . \tag{4.29}$$

If we take the solution

$$Q(R) = A \cdot e^{s/v_c R} + B \cdot e^{-s/v_c R} \quad , \tag{4.30}$$

then we find

$$U(R,t) = L^{-1}(AF_1 + BF_2) \tag{4.31}$$

and

$$\frac{\partial U}{\partial R} = L^{-1}(AF_3 + BF_4) \tag{4.32}$$

where

$$F_1 = \frac{e^{s/\sim_c R}}{R}\left(\frac{s}{\sim_c} - \frac{1}{R}\right), \qquad F_2 = \frac{e^{-s/\sim_c R}}{R}\left(\frac{-s}{\sim_c} - \frac{1}{R}\right)$$

$$F_3 = \frac{e^{s/\sim_c R}}{R}\left(\frac{s^2}{\sim_c^2} - \frac{2s}{\sim_c R} + \frac{2}{R^2}\right), \qquad F_4 = \frac{e^{-s/\sim_c R}}{R}\left(\frac{s^2}{\sim_c^2} + \frac{2s}{\sim_c R} + \frac{2}{R^2}\right). \tag{4.33}$$

In order that we consider only progressive waves, we take

$$U(R,t) = L^{-1}\left[B\, F_2\right] \tag{4.34}$$

and

$$T_{RR} = L^{-1}\left[(\lambda + 2\mu)\, BF_4 + \frac{2\lambda}{R} BF_2\right]. \tag{4.35}$$

If we take the pressure function in the form

$$p(t) = L^{-1}\left[f(s)\right] \tag{4.36}$$

then for $R = a$, some initial radius, set

$$T_{RR} = -p(t), \text{ for } t > 0,$$
$$= 0, \quad \text{ for } t < 0,$$

and so

$$-L^{-1}\left[f(s)\right] = L^{-1}\left[(\lambda + 2\mu)BF_4(s,a) + \frac{2\lambda}{a} BF_2(s,a)\right]$$

or

$$B = \frac{-f(s)}{(\lambda + 2\mu)F_4(s,a) + \frac{2\lambda}{a} F_2(s,a)} \tag{4.37}$$

hence, finally

$$U(R,t) = L^{-1}\left[\frac{-f(s)\ F_2(s,R)}{(\lambda+2\mu)F_4(s,a)+\frac{2\lambda}{a}F_2(s,a)}\right] \qquad t>0 \qquad (4.38)$$

$$= 0 \qquad\qquad t<0$$

In detail this solution is

$$U(R,t) = \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta}\frac{v_c a^3(Rs+v_c)\ e^{s\left(t-\frac{R-a}{v_c}\right)}\ f(s)\ ds}{R^2\left[(\lambda+2\mu)(as)^2+4\mu v_c(as)+4\mu v_c^2\right]}\ ,\quad t>0 \qquad (4.39)$$

$$=0\ ,\quad t<0.$$

We see that these are progressive waves out from the sphere. If we set

$$p(t) = L^{-1}\left[p_o/s\right]$$

then

$$\begin{aligned} p(t) &= p_o\ , & t&>0 \\ &= p_o/2\ , & t&=0 \\ &= 0\ , & t&<0\ . \end{aligned}$$

Using this form of f(s) in equation 4.39, we find for a unit pressure pulse,

$$U(R,t) = \frac{ap_o}{4\mu}\left[\left(\frac{a}{R}\right)^2\sqrt{\frac{2\mu+\lambda}{\mu+\lambda}}\left(\frac{a}{R}\right)^2 e^{-\frac{v_c}{a}\gamma_1\tau}\sin\left(\frac{v_c}{a}\gamma_2\tau+\tan^{-1}\frac{\gamma_2}{\gamma_1}\right)\right. \qquad (4.41)$$

$$\left. +\frac{2}{\sqrt{1+\lambda/\mu}}\left(\frac{a}{R}\right)\ e^{-\frac{v_c}{a}\gamma_1\tau}\sin\frac{v_c}{a}\gamma_2\tau\ ,\right] \qquad \tau>0$$

$$=0, \qquad \tau<0$$

where

$$\tau = t - \frac{(R-a)}{v_c}, \quad \gamma_1 = \frac{1}{1+\frac{\lambda}{2\mu}}, \quad \gamma_2 = \frac{\sqrt{1+\lambda/\mu}}{1+\frac{\lambda}{2\mu}} \tag{4.42}$$

and

$$\frac{\gamma_2}{\gamma_1} = \sqrt{1+\frac{\lambda}{\mu}} \tag{4.43}$$

We see that the total displacement consists of three terms, one static and two oscillatory. The first decaying as $1/R^2$ and the second as $1/R$. If we assume that a is small and R fairly large, then the dominant oscillatory term is

$$U(R,t) = \frac{a^2 p_o}{2\mu\sqrt{1+\lambda/\mu}} \cdot \frac{1}{R} \; e^{-v_c/a\,\gamma_1 \tau} \sin \frac{v_c}{a} \gamma \tau, \quad \tau \geq 0 \tag{4.44}$$

$$= 0, \quad \tau < 0 \; .$$

We observe amplitude $\propto$, pressure, area of cavity, 1/rigidity of medium, frequency $=(v_c\,\gamma_2\,/2\pi a)\propto$, velocity of wave propagation, 1/radius of cavity, and if $\lambda = \mu$, the damping is high, hence the motion is in the nature of a pulse of duration

$$\Delta t = \frac{3\pi}{2\sqrt{2}} \cdot \frac{a}{v_c}, \qquad f = \frac{\sqrt{2}}{3\pi} \cdot \frac{v_c}{a} \; . \tag{4.45}$$

4.4 ELASTIC HALF-SPACE.

From the spherical case, we have for any pressure time distribution at R=a, the radial displacement

$$U(R,t) = L^{-1}\left[BF_2\right] = L^{-1}\left[B \frac{\partial}{\partial R}\left(\frac{e^{-(s/v_c)R}}{R}\right)\right]. \tag{4.46}$$

In cylindrical coordinates, this becomes

$$u_o = \frac{U \cdot r}{R}, \qquad w_o = \frac{U(z+d)}{R} \tag{4.47}$$

for the spherical disturbance located at $z=-d$ with $R=\sqrt{r^2+(z+d)^2}$. For a spherical disturbance located at $z=d$, we have analogously

$$u_1=\frac{U'r}{R'},\qquad w_1=\frac{U'(z-d)}{R'} \tag{4.48}$$

where $R'=\sqrt{r^2+(z-d)^2}$ and $U'(R,t)$ is a solution like equation 4.46.

If we combine the two solutions, we have

$$\mu\left[\frac{\partial}{\partial z}(u_o+u_1)+\frac{\partial}{\partial r}(w_o+w_1)\right]=\tau_{rz}=0 \tag{4.49}$$

at $z=0$, or this means that we have zero shear stress from this sum at $z=0$. If we form the cylindrical components of displacement, we have

$$u_o=L^{-1}\left[B\,\frac{r}{R}\,\frac{\partial}{\partial R}\left(\frac{e}{R}\right)^{-(s/v_c)R}\right] \tag{4.50}$$

and

$$w_o=L^{-1}\left[B\,\frac{(z+d)}{R}\frac{\partial}{\partial R}\left(\frac{e}{R}\right)^{-(s/v_c)R}\right]. \tag{4.51}$$

Noting that

$$\frac{\partial}{\partial r}=\frac{r}{R}\frac{\partial}{\partial R},\qquad \frac{\partial}{\partial z}=\frac{(z+d)}{R}\frac{\partial}{\partial R} \tag{4.52}$$

equations 4.50 and 4.51 may be written

$$u_o=L^{-1}\left[B\,\frac{\partial}{\partial r}\left(\frac{e}{R}\right)^{-s/v_c\,R}\right] \tag{4.53}$$

and

$$w_o=L^{-1}\left[B\,\frac{\partial}{\partial z}\left(\frac{e}{R}^{-s/v_c\,R}\right)\right] \tag{4.54}$$

From Watson's Bessel Functions [17] using p.416, 13.47 equation (2) and p. 80, 3.71 equation (13), taking $\mu=0$, $\nu=\frac{1}{2}$, $a=(z+d)$, $b=r$ and $z=s/v_c$, we have

$$\frac{e^{-s/v_c R}}{R} = \int_0^\infty \frac{\eta}{\alpha} e^{-(z+d)\alpha} J_0(\eta r)\, d\eta, \quad (z+d) > 0 \tag{4.55}$$

where $\alpha = \sqrt{\eta^2 + s^2/v_c^2}$. This integral is clearly uniformly convergent. By differentiating under the integral sign, and using this in equations 4.53 and 4.54, we have

$$u_0 = L^{-1}\left[-B \int_0^\infty \frac{\eta^2}{\alpha} e^{-(z+d)\alpha} J_1(\eta r)\, d\eta \right] \tag{4.56}$$

and $\qquad (z+d) > 0$

$$w_0 = L^{-1}\left[-B \int_0^\infty \eta\, e^{-(z+d)\alpha} J_0(\eta r)\, d\eta \right] \tag{4.57}$$

For the image located at $z = d$, we have similarly

$$u_1 = L^{-1}\left[-B \int_0^\infty \frac{\eta^2}{\alpha} e^{-(d-z)\alpha} J_1(\eta r)\, d\eta \right] \tag{4.58}$$

and $\qquad (d-z) > 0$

$$w_1 = L^{-1}\left[-B \int_0^\infty \eta\, e^{-(d-z)\alpha} J_0(\eta r)\, d\eta \right] \tag{4.59}$$

where we have chosen B to be of the same form for equations 4.58 and 4.59 as in 4.56 and 4.57 in order that 4.49 be satisfied. In cylindrical coordinates, we must have zero vertical stress at z = 0. Explicitly, this requirement on the displacements is

$$\left(v_c^2 - 2v_s^2\right)\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + v_c^2 \frac{\partial w}{\partial z} = 0 \tag{4.60}$$

Let us take the displacements given by equations 4.56 through 4.59 and combine with

$$u_2 = L^{-1}\left[B \int_0^\infty \eta A(\eta) e^{\alpha z} J_1(\eta r)\, d\eta \right] \tag{4.61}$$

$$w_2 = L^{-1}\left[B \int_0^\infty \eta^2 C(\eta) e^{\beta z} J_0(\eta r)\, d\eta \right]$$

where $\alpha = \sqrt{\eta^2 + \frac{s^2}{v_c^2}}$, $\beta = \sqrt{\eta^2 + \frac{s^2}{v_s^2}}$ and A and C are functions

of η which we shall determine by equation 4.60 in order that we have zero shear stress and zero normal stress at $z=0$. Physically, this amounts to assuming that when the primary compressional disturbance, originating at the sphere, impinges on the surface $z=0$, it gives rise to waves of the compressional as well as shear type. Now we determine $A(\eta)$ and $C(\eta)$ so that

$$u = u_0 + u_1 + u_2 \quad , \quad w = w_0 + w_1 + w_2$$

are displacements which produce zero shear τ_{rz} and zero vertical stresses at $z=0$. Having done this, we obtain the formulae valid for $|z| < d$

$$u(r,z,t) = L^{-1}\Big[B \Big\{ \int_0^\infty \frac{\eta^2}{\alpha} \frac{f}{F} e^{\alpha(z-d)} J_1(\eta r)\, d\eta \qquad (4.62)$$

$$- \int_0^\infty \frac{\eta^2}{\alpha} e^{-\alpha(z+d)} J_1(\eta r)\, d\eta$$

$$-4\int_0^\infty \beta\eta^2 \frac{(2\eta^2 + s^2/v_s^2)}{F} e^{\beta z - \alpha d} J_1(\eta r)\, d\eta \Big\}\Big] \quad t > 0$$

$$= 0 \qquad t < 0$$

and

$$w(r,z,t) = L^{-1}\Big[B \Big\{ \int_0^\infty 4\eta^2 \frac{(2\eta^2 + s^2/v_s^2)}{F} e^{\beta z - \alpha d} J_0(\eta r)\, d\eta \qquad (4.63)$$

$$- \int_0^\infty \frac{f}{F} \eta e^{\alpha(z-d)} J_0(\eta r)\, d\eta$$

$$- \int_0^\infty \eta e^{-\alpha(z+d)} J_0(\eta r)\, d\eta \Big\}\Big] \quad t > 0$$

$$= 0 \qquad t < 0$$

where

$$F(\eta) = \left(2\eta^2 + \frac{s^2}{v_s^2}\right) - 4\alpha\beta\eta^2, \tag{4.64}$$

$$f(\eta) = \left(2\eta^2 + \frac{s^2}{v_s^2}\right) + 4\alpha\beta\eta^2, \tag{4.65}$$

$$\alpha = \sqrt{\eta^2 + s^2/v_c^2}$$

and

$$\beta = \sqrt{\eta^2 + s^2/v_s^2}\,.$$

We have assumed here that the initial function f(s) is so chosen as to represent some impulsive type of pressure on the spherical surface. The square roots are taken positive when η is large and positive. We can show by differentiation that these displacements are derivable from the stress functions

$$\Phi = \int_0^\infty \frac{2\eta}{\alpha} \cosh \alpha z \cdot e^{-\alpha d} J_0(\eta r)\, d\eta$$

$$- \int_0^\infty \frac{2\eta}{\alpha} \frac{\left(2\eta^2 + s^2/v_s^2\right)}{F(\eta)} e^{\alpha(z-d)} J_0(\eta r)\, d\eta \tag{4.66}$$

and (4.66)

$$\Psi = \int_0^\infty 4\eta \left(2\eta^2 + \frac{s^2}{v_s^2}\right) e^{\beta z - \alpha d} J_0(\eta r)\, d\eta. \tag{4.67}$$

Equations 4.62 and 4.63 give the exact solution for the displacements in an elastic half-space due to the application of an arbitrary radial pressure-time pulse on the surface of a small spherical cavity of radius a imbedded in the half-space a distance d from the surface. Previously, solutions have been given for a unit pulse in the elastic half-space considered as the instantaneous injection of a small volume in the medium. However, using this solution to build an arbitrary shaped pulse would require further integration procedures. With the forms given here an arbitrary type pulse can be initially given on the surface and is expressed in the operation $L^{-1}\left[B\{\cdots\cdots\}\right]$. Specifically, we have for the surface displacements, placing z=0,

$$u(r,0,t) = L^{-1}\left[B\left\{\int_0^\infty 2\beta\left(\frac{-2s^2\eta}{v_s^2F(\eta)}\right)e^{-\alpha d}J_1(\eta r)\,d\eta\right\}\right],\quad t>0$$

$$= 0 \qquad t<0 \tag{4.68}$$

and

$$w(r,0,t) = L^{-1}\left[B\left\{\int_0^\infty\left(2\eta^2+\frac{s^2}{v_s^2}\right)\left(\frac{-2s^2\eta}{v_s^2F(\eta)}\right)e^{-\alpha d}J_0(\eta r)\,d\eta\right\}\right] \quad t>0 \tag{4.69}$$

$$= 0 \qquad t<0$$

These may be written with only $J_0(\eta r)$ appearing as

$$u(r,0,t) = L^{-1}\left[B\left\{\frac{d}{dr}\int_0^\infty\frac{-2\beta}{\eta}\left(\frac{-2s^2\eta}{v_s^2F(\eta)}\right)e^{-\alpha d}J_0(\eta r)\,d\eta\right\}\right],\quad t>0 \tag{4.70}$$

$$= 0 \qquad ,\ t<0$$

and

$$w(r,0,t) = L^{-1}\left[B\left\{\int_0^\infty\left(2\eta^2+\frac{s^2}{v_s^2}\right)\left(\frac{-2s^2\eta}{v_s^2F(\eta)}\right)e^{-\alpha d}J_0(\eta r)\,d\eta\right\}\right],\quad t>0 \tag{4.71}$$

$$= 0 \qquad ,\ t<0$$

In these expressions

$$F(\eta) = \left(2\eta^2+\frac{s^2}{v_s^2}\right)^2 - 4\sqrt{\eta^2+\frac{s^2}{v_c^2}}\cdot\sqrt{\eta^2+\frac{s^2}{v_s^2}}\ \eta^2 \tag{4.72}$$

which has branch points at $\eta_0 = is/v_c$, $\eta_1 = is/v_s$ and a pole $\eta_3^2 = (-s^2/v_s^2)(1.08766)$ for $\lambda = \mu$. Lamb[8] has shown for

$$G(m) = (2m^2-k^2)^2 - 4\sqrt{m^2-h^2}\cdot\sqrt{m^2-h^2}\cdot m^2.$$

that $G(m)$ is a cubic in m^2/k^2 and if h and k are real then we have one essential real root and two extraneous complex conjugates. These last two roots make no contribution. If we write

$$G(im) = (2m^2+k^2)^2 - 4\sqrt{m^2+k^2}\cdot\sqrt{m^2+h^2}\cdot m^2$$

then for the case on hand, we identify $k = s/v_c$, $h = s/v_c$, $im = \eta$ and $F(\eta) = G(im)$. We have therefore to consider the evaluation of the integrals around a suitably chosen con-

tour enclosing the three singularities. From this integration with respect to η, we obtain functions of s and r only when we are at the surface $z=0$. Then by applying $L^{-1}[B\{\cdots\}]$ we introduce whatever type of pulse we would consider. Evaluation of these integrals yet remains and this is to be done at some future time.

CHAPTER 5

A CLASS OF CENTRAL FORCE MODELS IN STATISTICAL MECHANICS

5.1 INTRODUCTION

The object of this chapter is to study carefully the particular class of models in statistical mechanics described in § 5.2 with a view to deriving the equations of the mass-motion of the particles of the system considered as a continuous medium. The resulting equations constitute a sort of theory of hydrodynamics corresponding to the given model.

Since a finite system of particles is not a "continuour medium" some way must be found to pass from the discrete to the continuous. The method of this chapter is to consider an infinite sequence of particle distributions in which the number N of particles becomes infinite. Of course as N is varied, the laws of force acting between the particles must be adjusted in order to preserve the important characteristics of the system; this is done by choosing the potential $\Phi_N(r)$ (r = distance) of the N-th force law to satisfy

$$m_N \, \Phi_N(r) = K \, \Phi(r/\sigma_N)$$

where K is a constant having the dimensions of the square of velocity and

$$M = Nm_N \, , \quad m_N = D \, \sigma_N^3 \, ,$$

M being the total mass, m_N the mass of a single particle, D a constant having the dimensions of a density, σ_N a scale length, and $\Phi(\rho)$ is a fixed function, K,M,D, and Φ being independent of N.

The particle distributions are handled by considering the Fourier-Stieltjes transforms of the distributions of mass, momentum, and energy which transforms have desirable continuity and differentiability properties in the transformed y-space. Some very interesting theorems concerning the existence of the limits (as $N \to \infty$) of these transforms which vary continuously with time also are proved in §5.4.

The remainder of the chapter is devoted to the determination of the equations satisfied by the limiting distributions. These are seen to be the Fourier transforms of the desired equations which are as follows:

$$\rho_t + \overline{u}^{\alpha} \rho_{x^\alpha} = 0$$

$$\overline{u}^{\alpha}_t + \overline{u}^{\beta}\, \overline{u}^{\alpha}_{x^\beta} + p_{x^\alpha} = 0, \qquad \alpha = 1,2,3$$

$$\varepsilon_t + \overline{u}^{\alpha}\, \varepsilon_{x^\alpha} + \rho^{-1}(q^{\alpha}_{x^\alpha} + p\overline{u}^{\alpha}_{x^\alpha}) = 0$$

in which a repeated Greek index in a term indicates a summation of all the terms obtained by letting the index run from 1 to 3 (see also § 2.2; we use this summation convention for Greek indices throughout this chapter), ρ is the density, $\overline{u}^1, \overline{u}^2$, and $\overline{u}^3$ are the components of mass-velocity, ε is the internal energy per unit mass, p is the pressure, q^1, q^2, and q^3 are the components of the heat flux vector, all being functions of the time t and the rectangular coordinates x^1, x^2, x^3. The following relations also define p and q^{α} in terms of the other functions ρ, $\overline{u}^{\alpha}$, and ε and the constant K and the function Φ entering into the force laws of the model:

$$p = (\rho/2)\left\{4\left[\varepsilon - \beta(\rho)\right]/3 + K\rho C(\varepsilon/K)/D\right\}$$

$$q^{\alpha} = (\rho\overline{u}^{\alpha}/2)\left\{5\left[\varepsilon - \beta(\rho)\right]/2 + K\rho A(\varepsilon/K)/D + K\rho C(\varepsilon/K)/D - 2p/\rho - 2\varepsilon\right\}$$

where $A(s)$, $C(s)$, and $\beta(\rho)$ are functions determined by the force law and

$$A(s) = 4\pi\int_0^{\infty} w^2\,\Phi(w)\,\exp\left[-3s\,\Phi(w)/2\right] dw,$$

$$C(s) = (4\pi/3)\int_0^{\infty} -w^3\,\Phi'(w)\,\exp\left[-3s\,\Phi(w)/2\right] dw;$$

we have not determined the explicit form of the function $\beta(\rho)$ but believe that it is given, at least for convex functions Φ, by

$$\beta(\rho) = K \sum_{j,k,\ell=-\infty}^{\infty} \Phi\left[(j^2+k^2+\ell^2+jk+j\ell+k\ell)^{\frac{1}{2}}(3FD/4\pi\rho)^{\frac{1}{3}}\right]$$

where F is the numerical density of the densest packing of spheres, i.e.,

$$F = \sqrt{2}\left[3 \text{ Arc } \cos(1/3)-\pi\right], \quad 3F/4\pi = .186128.$$

We have also determined the form of the density function for the distribution of coordinates and velocities as follows:

$$\pi(t;x,u) = \rho(t;x)\cdot\left[B(\varepsilon,\rho)/\pi\right]^{3/2} \exp\left[-B(\varepsilon,\rho)\left|u-\bar{u}(t;x)\right|^2\right]$$

where

$$B(\varepsilon,\rho) = 3/4\left[\varepsilon-\beta(\rho)\right].$$

We note that

$$\int_{-\infty}^{\infty}\pi(t;x,u)du = \rho(t;x), \quad \int_{-\infty}^{\infty}u^{\alpha}\pi(t;x,u)du = \rho\bar{u}^{\alpha}.$$

The explicit forms of the functions p, q^{α}, A(s), C(s), and π were obtained by making a series of assumptions which are set forth and underlined at various points in the derivation. Most of these assumptions are in the nature of approximations which the writer believes are valid in the limit; unfortunately, at this writing, the writer has not investigated them carefully. However, we have made a fundamental assumption, stated as Assumption 5 in §5.5 which is closely allied with the famous Ergodic Hypothesis. It is probable that this will not be proved by anybody in the foreseeable future; the best that can be hoped for is to supply a good deal of heuristic evidence in its support.

The equations which we have obtained are the standard ones for liquids and gases, although the writer believes that the explicit determinations of the "equations of state" are new. The writer believes that the solid or perhaps plastic states correspond to cases where ε is very close to its lower limit $\beta(\rho)$. For notice what happens to the function π in such cases; this indicates that the random motions of the particles with respect to their mass-velocity become very small. This in turn points to a definite breakdown of assumption 5 on account of the inability of the individual particles to change their relative arrangement. This is an exceedingly interesting line of investigation which we wish to pursue further. We believe also that our

Assumption 1 on the nature of the function Φ rules out the possibility of a liquid state for our model. There are no terms which involve viscosity in our equations; accordingly, if our approximations are valid the viscosity effects arise from the finite size of the particles so their determination will require a further investigation of the approximations. Some of these involve the neglect of terms of the order of σ_N and σ_N^2 which might be found without too much trouble; however, so many of them involve terms of the order of σ_N^3 that it is probably hopeless (and also unnecessary) to carry an expansion in terms of σ_N to terms beyond the second power.

We have already alluded to the summation convention with respect to repeated Greek indices which we shall employ throughout this chapter. We shall make extensive use of superscripts and subscripts, and the superscripts will be located in the places usually occupied by exponents. Superscripts will sometimes denote exponents and such cases will usually be clear from the context, as in $(B/\pi)^{3/2}$, etc., and in connection with the exponential function which we write alternatively as

$$e^z \text{ or } \exp z$$

the latter being used if z is some complicated expression. We shall also use double subscripts or superscripts such as κ_{pq}, etc.; a comma is to be understood between the subscripts or superscripts and will be inserted if either p or q is complicated.

We shall frequently use single letters, possibly with <u>subscripts</u>, to denote <u>vectors</u>, the components of a vector (in 3 space) will be designated by <u>superscripts</u>; thus x_j^α denotes the α-th component of the j-th vector. The inner product of two vectors y and x will be denoted by

$$y \cdot x = y^\alpha x^\alpha$$

and the length of a vector w by $|w|$. We shall write

$$f(x_1,\cdots,x_N) \quad \text{for} \quad f(x_1^1,x_1^2,x_1^3,\cdots,x_N^1,x_N^2,x_N^3), \text{ etc.}$$

We shall use the term <u>cell</u> (in 3-space) to denote the set of all (x^1,x^2,x^3) (or some other letter) satisfying

$$a^\alpha \leq x^\alpha < b^\alpha, \quad \alpha = 1,2,3,$$

for fixed numbers a^α and b^α and will denote cells by the letter R, possibly with subscripts. Cells in 6-space are defined similarly. Parts of the boundaries have been left off so that cells can be fitted together without counting boundary points several times (particles on the common boundaries of several cells otherwise would be counted in all these cells).

Multiple integrals will frequently occur and will be denoted by a single integral sign except where it is desired to express a multiple integral as the result of several repeated (possibly multiple) integrals. Thus an expression such as

$$\int_{S_1} dx_1 \int_{S_2} dx_2 \int_{S_3} h(x_1,x_2,u_1)du_1$$

will denote the result of integrating h first with respect to the variables in u_1 (each letter may denote several variables) over the domain of integration S_3 (which might depend on x_1 and x_2) holding x_1 and x_2 constant, then integrating that result with respect to x_2 over S_2, and last with respect with respect to x_1 over S_1.

A function f of several variables, say $(t;y) = (t;y^1,y^2,y^3)$, will be said to <u>satisfy a Lipschitz condition</u> on a set S if there is a constant L such that

$$\left|f(t_1;y_1)-f(t_2;y_2)\right| \leq L\left[(t_2-t_1)^2+|y_2-y_1|^2\right]^{\frac{1}{2}}$$

for any two points $(t_1;y_1)$ and $(t_2;y_2)$ on S. If S is a cell (or the whole space, etc.) and the partial derivatives are continuous and uniformly bounded on S, then f satisfies such a condition but the converse is not necessarily true.

5.2 PARTICLE DISTRIBUTIONS

The models which we consider in this chapter consist of N identical particles, each of mass m, any two of which repel one another with a force

$$-m^2\, \Phi'(r)$$

(if $\Phi'(r) > 0$ the particles attract one another). Let

$$H(x_1,\cdots,x_N) = \frac{m}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N} \Phi(r_{jk}),\quad r_{jk} = |x_j - x_k|. \tag{5.1}$$

Then the equations of motion of the system are

$$m\ddot{x}_j^\alpha = -m\,\partial H/\partial x_j^\alpha\ ,\quad \alpha = 1,2,3,\quad j = 1,\cdots,N. \tag{5.2}$$

or

$$\dot{x}_j^\alpha = u_j^\alpha \quad\text{and}\quad \dot{u}_j^\alpha = -\,\partial H/\partial x_j^\alpha\ , \tag{5.3}$$

where dots denote differentiation with respect to time. We may think of these equations as determining the motion of a single particle of mass

$$M = Nm$$

in the 6N-dimensional space of the vectors $x_1,\cdots,x_N$ and $u_1,\cdots,u_N$. This space is called the phase space and the particle is called the phase particle.

From equations 5.3, it follows that all the quantities

$$m\sum_{j=1}^{N} u_j^\alpha\ ,\quad m\sum_{j=1}^{N}(x_j^\alpha - tu_j^\alpha),\quad m\sum_{j=1}^{N}(x_j^\alpha u_j^\beta - x_j^\beta u_j^\alpha) \tag{5.4}$$

$$\frac{m}{2}\sum_{j=1}^{N}|u_j|^2 + mH \tag{5.5}$$

are constant in time along each trajectory of a phase particle. The first and third quantities in equation 5.4 give the components of the total momentum and angular momentum (about the origin) of the system of particles and the first and second terms in equation 5.5 are the total kinetic and potential energies of the system, respectively. The three components of angular momentum are obtained by setting $(\alpha,\beta) = (2,3),(3,1)$, and $(1,2)$.

We are interested in the distributions over the x-space of mass and the other quantities mentioned above. These are determined by a knowledge of the total mass, momentum, etc., of all the particles in each cell R of the x space. The total mass in R at a given instant of time is just m times the number of particles in R at that instant and the components of momentum and angular momentum and the kinetic

energy are given by the sums

$$m \sum_j u_j^\alpha, \; m \sum_j (x_j^\alpha u_j^\beta - x_j^\beta u_j^\alpha), \; \frac{m}{2} \sum_j |u_j|^2 \qquad (5.6)$$

where the sums are extended over the particles x_j which are in R at that instant. It is customary to define the potential energy of the single j-th particle by

$$\frac{m^2}{2} \sum_{\substack{k=1 \\ k\neq j}}^{N} \Phi(r_{jk}); \qquad (5.7)$$

this merely assumes that the potential energy between two particles is shared equally between them. It is seen that the total potential energy of the system is then just the sum of that of all the particles and so we define the potential energy of the particles in R as the sum of their separate potential energies.

Obviously these quantities for any fixed cell R vary discontinuously with time as particles enter and leave R. Moreover, it is obvious that these distributions are not integrals over R of continuous functions of x. In order to derive equations for the density, mass-velocity (or momentum-density), and local energy, considered as continuous functions of t and x, we must somehow pass from the "discrete" particle distributions just described to "continuous" distributions in which the mass, momentum, etc., in a cell R are triple integrals over R of density, etc., all of which are continuous and have continuous first derivatives with respect to t and the x^α.

This is frequently done by considering a continuous family of particle-systems, in other words a continuous family of phase particles, and then introducing a weighted average in the phase-space over this set of phase particles. This introduces such a degree of arbitrariness into the situation that it is difficult to draw conclusions of physical significance. We shall study particle distributions by means of their Fourier-Stieltjes transforms which turn out to have desirable continuity and differentiability properties. Our method of passing from the discrete to the continuous consists in considering sequences of particle distributions in which the number N of particles becomes infinite. A very general theorem concerning the existence of limiting distributions is proved in §5.4. We then present a heuristic argument, based on certain assumptions which are set forth in the course

of the argument, which leads to the form of the equations satisfied by the limiting distributions.

In Statistical Mechanics, it is frequently desirable to consider the distributions of the quantities mentioned above in the 6 dimensional (x,u) space. These are defined by equations 5.6 and 5.7 where the sums extend over all j for which (x_j,u_j) is in the cell R in the (x,u) space; in other words for all j for which we have simultaneously x_j in R_1 and u_j in R_2, R_1 and R_2 being the projections of R on the x and u space, respectively. If we denote the total mass in such cells R by $\Pi(R)$, then Π is called the simultaneous distribution of coordinates and velocities. The distributions in the (x,u) space of momentum, angular momentum, and kinetic energy can be expressed formally in terms of the distribution Π by means of the Stieltjes integrals

$$\int_R u^{\alpha} d\Pi, \quad \int_R (x^{\alpha}u^{\beta}-x^{\beta}u^{\alpha})d\Pi, \quad \text{and} \quad \tfrac{1}{2}\int_R |u|^2 d\Pi \ ;$$

the potential energy cannot be so expressed. In case Π were a "continuous distribution", i.e. if there were a continuous function $\pi(t;x,u)$ such that

$$\Pi(R) = \int_R \pi(t;x,u)dxdu$$

all the other distributions would be continuous, reducing to

$$\int_R u^{\alpha}\pi(t;x,u)dxdu, \quad \int_R (x^{\alpha}u^{\beta}-x^{\beta}u^{\alpha})\pi(t;x,u)dxdu, \text{ and}$$

$$\tfrac{1}{2}\int_R |u|^2\pi(t;x,u)dxdu,$$

respectively. The distribution Π is therefore also of considerable interest.

5.3 SEQUENCES OF PARTICLE DISTRIBUTIONS; A THEOREM ON BOUNDEDNESS

We wish now to consider sequences of particle distributions in which N is allowed to vary and we shall wish to allow N (which is very large anyway) to tend to infinity. In order for there to be limiting distributions of a reasonable sort, we shall assume that the total mass, momentum components, and energy remain constant. Thus we must attach a subscript

N to m and Φ and must have

$$Nm_N = M \qquad (5.8)$$

independently of N. It is convenient to introduce a distance scale factor σ_N having the dimensions of a length defined by

$$m_N = D\,\sigma_N^3 \qquad (5.9)$$

where D has the dimensions of density. Since we are interested in systems in which the potential energy is important (dense gases, liquids, and solids), we need to choose the form of Φ_N so that the potential energy term tends to a limit as $N \to \infty$. We choose Φ_N so that

$$m_N\,\Phi_N(r) = K\,\Phi(r/\sigma_N)$$

where K must have the dimensions of the square of velocity and Φ is a fixed function independent of N.

The equations of motion then become

$$\dot{x}_j^a = u_j^a \text{ and } \dot{u}_j^a = -K\,\sigma_N^{-1} \sum_{\substack{k=1 \\ k \neq j}}^{N} \Phi'(r_{jk}/\sigma_N)(x_j^a - x_k^a)/r_{jk}. \qquad (5.10)$$

The components of total momentum and the total energy become

$$M \cdot \frac{1}{N} \sum_{j=1}^{N} u_j^a \text{ and } M \cdot \frac{1}{2N} \sum_{j=1}^{N} \left[|u_j|^2 + K \sum_{\substack{k=1 \\ k \neq j}}^{N} \Phi(r_{jk}/\sigma_N) \right] \qquad (5.11)$$

and these remain constant with time. We note here the presence of the factor σ_N^{-1} in the expression for $\dot{u}_j^a$. From equations 5.8 and 5.9 we see that $\sigma_N \to 0$ as $N \to \infty$. This suggests that the u-components of the motion of the phase particle vary more and more rapidly as $N \to \infty$. It also raised the question as to what quantities remain bounded and what quantities have bounded time derivatives as $N \to \infty$. In this connection we first prove the following theorem:

Theorem 5.1 Suppose $\Phi(\rho)$ is continuous and differentiable for all $\rho > 0$ with

$$\Phi(\rho) \geqq 0 \quad \text{and} \quad \Phi'(\rho) \leqq 0 \quad \text{if} \quad \rho > 0, \qquad (5.12)$$

and suppose there is a number $n > 0$ such that

$$0 \leqq -\rho\,\Phi'(\rho) \leqq n\,\Phi(\rho) \quad \text{for} \quad \rho > 0. \qquad (5.13)$$

Suppose a particle distribution has total energy E and suppose at some instant t_o, we have

$$M \cdot \frac{1}{N} \sum_{j=1}^{N} |x_j|^2 = C < \infty .$$

Then, for all times we have

$$M \cdot \frac{1}{N} \sum_{j=1}^{N} |x_j|^2 \leqq C + 2^{3/2}(CE)^{1/2}|t-t_o| + n'E \cdot |t-t_o|^2 \qquad (5.14)$$

where

$$n' = \text{the larger of 2 and } n. \qquad (5.15)$$

Proof. If we let $f(t)$ denote the left side of equation 5.14, then

$$f'(t) = 2M \cdot \frac{1}{N} \sum_{j=1}^{N} (x_j \cdot u_j),$$

$$f''(t) = 2M \cdot \frac{1}{N} \sum_{j=1}^{N} |u_j|^2 + 2M \cdot \frac{1}{N} \sum_{j=1}^{N} x_j^{a} \dot{u}_j^{a}$$

using the equations of motion 5.10. From the equations of motion, we obtain

$$2M \cdot \frac{1}{N} \sum_{j=1}^{N} x_j^{a} \dot{u}_j^{a} = 2MK\,\sigma_N^{-1} \cdot \frac{1}{N} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} x_j^{a} \left[-\Phi'(r_{jk}/\sigma_N) \cdot (x_j^{a} - x_k^{a})/r_{jk} \right].$$

Noting that the interchange of j and k in this double sum changes the sign of the term in brackets, we may add a duplicate of this sum with j and k interchanged and divide the result by 2 obtaining

$$2M\cdot\frac{1}{N}\sum_{j=1}^{N}x_j^{\alpha}u_j^{\alpha}=MK\,\sigma_N^{-1}\cdot\frac{1}{N}\sum_{\substack{j,k=1\\j\neq k}}^{N}(x_j^{\alpha}-x_k^{\alpha})\left[-\Phi'(r_{jk}/\sigma_N)\cdot(x_j^{\alpha}-x_k^{\alpha})/r_{jk}\right]$$

$$=MK\cdot\frac{1}{N}\sum_{\substack{j,k=1\\j\neq k}}^{N}\left[-(r_{jk}/\sigma_N)\;\Phi'(r_{jk}/\sigma_N)\right].$$

Using equations 5.12, 5.13, and 5.15, we see that

$$0\leqq f''(t)\leqq 4\cdot M\cdot\frac{1}{2N}\sum_{j=1}^{N}|u_j|^2+2n\cdot MK\cdot\frac{1}{2N}\sum_{\substack{j,k=1\\j\neq k}}^{N}\Phi(r_{jk}/\sigma_N)$$

$$\leqq 2n'E,$$

for all t. By the Schwartz Inequality

$$|f'(t_0)|=2M\cdot\left|\frac{1}{N}\sum_{j=1}^{N}(x_j\cdot u_j)\right|\leqq 2\left[M\,N^{-1}\sum|x_j|^2\right]^{\frac{1}{2}}\cdot\left[MN^{-1}\sum|u_j|^2\right]^{\frac{1}{2}}$$

$$\leqq 2^{3/2}(CE)^{\frac{1}{2}}.$$

The result follows.

From this result, it follows that all the "second moments" of the distribution π of coordinates and velocities, such as

$$M\cdot\frac{1}{N}\sum_{j=1}^{N}x_j^{\alpha}x_j^{\beta},\quad M\cdot\frac{1}{N}\sum_{j=1}^{N}x_j^{\alpha}u_j^{\beta},\quad\text{and}\quad M\cdot\frac{1}{N}\sum_{j=1}^{N}u_j^{\alpha}u_j^{\beta}$$

remain bounded on any finite time interval.

We remark that any function $\Phi(\rho)$ of the form

$$\Phi(\rho) = A\rho^{-n} \quad , \; n > 0$$

satisfies the conditions of this theorem. Many other functions do as well. No doubt one can obtain a boundedness theorem for a much more general class of functions Φ including some which change sign but this question requires further study.

5.4 THE FOURIER-STIELTJES TRANSFORMS; LIMIT THEOREMS

In this section, we introduce the Fourier-Stieltjes transforms of the distributions of mass, momentum, and total energy over the x^a-space. These are the complex-valued functions of t and y^a defined by

$$\psi^1(t;y) = M\cdot \frac{1}{N} \sum_{j=1}^{N} \exp\left[i(y\cdot x_j)\right] \qquad (i^2 = -1)$$

$$\psi^{1+a}(t;y) = M\cdot \frac{1}{N} \sum_{j=1}^{N} u^a\cdot\exp\left[i(y\cdot x_j)\right] \; , \; a = 1,2,3, \qquad (5.16)$$

$$\psi^5(t;y) = M\cdot \frac{1}{2N} \sum_{j=1}^{N} \left[|u_j|^2 + K \sum_{\substack{k=1 \\ k\neq j}}^{N} \Phi(r_{jk}/\sigma_N)\right] \exp\left[i(y\cdot x_j)\right]$$

The Fourier-Stieltjes transform of the distribution Π of coordinates and velocities is defined by

$$\varphi(t;y;v) = M\cdot\frac{1}{N} \sum_{j=1}^{N} \exp\left[i(y\cdot x_j \;\; v\cdot u_j)\right] = \int_{-\infty}^{\infty} \exp\left[i(y\cdot x \;\; v\cdot u)\right] d\Pi(x, \;)$$

(5.17)

We notice that the partial derivatives of φ are continuous and given by

$$\varphi_{y^a} = iM\cdot\frac{1}{N} \sum x_j^a \exp\left[i(y\cdot x_j + v\cdot u_j)\right]$$

$$\varphi_{v^a} = iM\cdot\frac{1}{N} \sum u_j^a \exp\left[i(y\cdot x_j + v\cdot u_j)\right]$$

$$\varphi_{y^\alpha y^\beta} = -M\cdot\frac{1}{N} \sum x_j^\alpha x_j^\beta \exp\left[i(y\cdot x_j + v\cdot u_j)\right] \tag{5.18}$$

$$\varphi_{y^\alpha v^\beta} = -M\cdot\frac{1}{N} \sum x_j^\alpha u_j^\beta \exp\left[i(y\cdot x_j + v\cdot u_j)\right]$$

$$\varphi_{v^\alpha v^\beta} = -M\cdot\frac{1}{N} \sum u_j^\alpha u_j^\beta \exp\left[i(y\cdot x_j + v\cdot u_j)\right]$$

From these formulas, we see also that

$$\varphi(t;y,0) = \psi^1(t;y)\ ,\quad \varphi_{v^\alpha}(t;y,0) = i\,\psi^{1+\alpha}(t;y),\ \alpha=1,2,3$$

We note that the functions $\psi^1,\ldots,\psi^5$ are differentiable with respect to t as follows:

$$\psi^1_t(t;y) = iy^\alpha\,\psi^{1+\alpha}(t;y)$$

$$\psi^{1+\alpha}_t(t;y) = Miy^\beta N^{-1} \sum_{j=1}^{N} u_j^\alpha u_j^\beta \exp\left[i(y\cdot x_j)\right]$$

$$+ (iy^\beta/2N)\sum_{j=1}^{N} \exp\left[i(y\cdot x_j)\right] \sum_{\substack{k=1\\k\neq j}}^{N} E\left[iy\cdot(x_k-x_j)\right] v_{jk}^{\alpha\beta} \tag{5.19}$$

$$\psi^5_t(t;y) = (iy^\alpha/N) \sum_{j=1}^{N} u_j^\alpha e_j \exp\left[i(y\cdot x_j)\right]$$

$$+(iy^\beta/2N)\sum_{j=1}^{N} u_j^\alpha \exp\left[i(y\cdot x_j)\right] \sum_{\substack{k=1\\k\neq j}}^{N} E\left[iy\cdot(x_k-x_j)\right] v_{jk}^{\alpha\beta}$$

where

$$e_j = (M/2)\left[\left|u_j\right|^2 + K \sum_{\substack{k=1\\k\neq j}}^{N} \Phi(r_{jk}/\sigma_N)\right]$$

$$v_{jk}^{\alpha\beta} = MK\left[-(r_{jk}/\sigma_N)\ \Phi'(r_{jk}/\sigma_N)\cdot(x_j^{\alpha}-x_k^{\alpha})(x_j^{\beta}-x_k^{\beta})/r_{jk}^{2}\right] \qquad (5.20)$$

$E(z) = z^{-1}\left[\exp z - 1\right]$ if $z\neq 0$, $E(0)=1$.

We note that σ_N enters into the derivatives of the in equations 5.19 and 5.20 only inside Φ or in the combination

$$-\rho\,\Phi'(\rho) \text{ with } \rho = r_{jk}/\sigma_N\ .$$

Suppose now that Φ satisfies the conditions in equations 5.12 and 5.13 and suppose we have any sequence of particle distributions in which $N\to\infty$, the total energies E_N remain bounded as does the quantity on the left side of equation 5.14 at some instant of time, all the bounds being independent of N. Then, since $|\exp(i\theta)|=1$ for all real θ, it follows from theorem 5.1 and equations 5.19 and 5.20 that Ψ_N^1 to Ψ_N^4 and their first derivatives with respect to t and the y^{α} are uniformly bounded independently of N for all y and all t on any finite interval. Hence we have the following theorem as an immediate consequence of Ascoli's theorem:

Theorem 5.2 Suppose we are given a sequence of particle distributions of the type described above. Then there is an infinite subsequence of the given sequence of N such that the functions Ψ_N^{γ} tend uniformly on any bounded part of (t,y) space to limiting functions Ψ^{γ}, $\gamma = 1,\cdots,4$, each of which satisfies a uniform Lipschitz condition on any bounded part of the (t,y) space. The function Ψ' is continuously differentiable with respect to time and satisfies the first equation in 5.19.

The derivative of $\Psi_N^{(5)}$ involves third moments and strange cross moments which we have not proved to be bounded in time. However if the energy distributions tend to zero uniformly at infinity and uniformly on any finite time interval (something which seems very likely if it holds at one instant), then the functions $\Psi_N^{(5)}$ are equi-continuous over the y-space anyway. Most probably there are sequences of particle distributions in which the required additional moments are bounded in time which would allow us to include Ψ^5 in the theorem above. The second derivatives of the $\Psi_N^{(\gamma)}$ $(\gamma = 2,\cdots,5)$ are seen to involve the factor σ_N^{-1} which suggests that the first derivatives of the $\Psi_N^{(\gamma)}$, though bounded, oscillate more and more rapidly with respect to time as N increases. Thus

theorem 5.2 is interesting in that it shows that for any time interval, however small, the time averages of the time derivatives of the $\psi_N^{(\gamma)}$ tend to limits, namely the difference quotients of the limit functions.

It is also found that the first time derivative of $\varphi(t;y,v)$ for $v \neq 0$ also contains the factor σ_N^{-1} so that also varies more and more rapidly as a function of time as N increases. But if we have a sequence of particle distributions as above we note (from equations 5.18) at least that the φ_N and their partial derivatives up to the second order in the y's and v's are uniformly bounded over the (y,v) space on any bounded time interval independently of N. Let us consider the functions

$$X_N(t;y,v) = \int_{t_0}^{t} \varphi_N(s;y,v)\, ds.$$

Then the functions X_N, X_{Ny^α}, and X_{Nv^α} and their derivatives with respect to t, y^β, and v^β are all uniformly bounded on any bounded part of (t,y,v) space. Hence we obtain the following theorem:

Theorem 5.3 Suppose we are given a sequence of particle distributions as in theorem 5.2. Then there is a subsequence of N such that X_N, X_{Ny^α}, and X_{Nv^α} all converge uniformly to limiting functions X, X_{y^α}, and X_{v^α} on any finite part of (t,y,v) space; the limiting functions satisfy uniform Lipschitz conditions in (t;y,v) on any finite part of (t;y,v) space.

The interest of this theorem lies in the observation that

$$\frac{X_N(t+\tau;y,v)-X_N(t;y,v)}{\tau} = \frac{1}{\tau}\int_t^{t+\tau} \varphi_N(s;y,v)\, ds,$$

i.e. is a time average of φ_N. Thus we conclude from the theorem that there is a subsequence of N such that the time averages over every time interval however short tend to limits. Finally, since the limit functions X, X_{y^α}, and X_{v^α} all satisfy Lipschitz conditions, it can be shown that there is a set of measure zero of values of t such that if t_1 is not in this set, then $X_t(t_1,y,v)$ exists for all (y,v) simultaneously and satisfies a Lipschitz condition in (y,v). Since t does not enter into the equations of motion, one would expect that X_t would exist and be continuous for all t but this has not been proved. In this case, we would call the corresponding distributions of

coordinates and velocities quasi-stable.

5.5 DETERMINATION OF THE LIMITING EQUATIONS; THE FUNDAMENTAL ASSUMPTION.

In this and following sections we wish to determine the forms of the limiting functions obtained in theorems 5.2 and 5.3 obtained in the preceding section.

We now make the following assumptions:

Assumption 1: Φ satisfies the conditions in equations 5.12 and 5.13.

Assumption 2: We are given a sequence of particle distributions such that their energies E_N and the quantities C_N on the left side of equations 5.14 are bounded and independently of N at some instant.

Assumption 3: The particle distributions are such that the functions $\psi_N^1, \ldots, \psi_N^5$ are all uniformly bounded, have uniformly bounded first derivatives, and converge uniformly to functions $\psi^1, \ldots, \psi^5$ on each bounded part of (t;y) space as in theorem 5.2.

The assumptions 2 and 3 above can always be satisfied, except possibly that about ψ^5, not yet proved. The next assumption reflects our desire to obtain equations governing the distributions in the x-space.

Assumption 4: The functions $\psi^1, \ldots, \psi^5$ have continuous first derivatives and are the Fourier-Stieltjes transforms of continuous distributions in the x-space the density functions of which have continuous derivatives.

For each N in our sequence, let D_N^γ (t;R) be the distribution corresponding to ψ_N^γ (t;y) and let D^γ(t;R) correspond to ψ^γ(t;y). From the theory of Fourier-Stieltjes transforms it follows that the distributions are uniquely determined and that the convergence of D_N^γ (t;R) to D^γ(t;R) is uniform for all cells and for all t on any finite interval. The distributions D' and D'_N are those of mass, the $D^{1+\alpha}$ and $D_N^{1+\alpha}$ are those of momentum, and D^5 and D_N^5 are those of energy. By assumption 4, and equations 5.16,

$$D^1(t;R) = \int_R \rho(t;x)dx, \quad D^{1+a}(t;R) = \int_R \rho(t,x)\overline{u}^a(t;x)dx,$$

$$D^5(t;R) = \int_R e(t;x)dx,$$

$e(t;x)$ being the total energy function and $\overline{u}^a(t,x)$ being the components of the mass velocity vector; also

$$\psi^1(t;y) = \int_{-\infty}^{\infty} \rho(t;x)\exp\left[i(y\cdot x)\right] dx,$$

$$\psi^{1+a}(t;y) = \int_{-\infty}^{\infty} \rho(t;x)\overline{u}^a(t;x)\exp\left[i(y\cdot x)\right] dx$$

$$\psi^5(t;y) = \int_{-\infty}^{\infty} e(t;x)\exp\left[i(y\cdot x)\right] dx$$

For each N, choose a finite number $R_1,\cdots,R_P$, $P = P_N$, of non-overlapping cells which together contain all the particles of the distribution D'_N. We assume that $P_N \to \infty$ and the diameter of each cell $\to 0$ as $N \to \infty$ but so slowly that

$$\lim_{N\to\infty} \sigma_N^3/\mu(R_N) = 0, \quad \lim_{N\to\infty} D_N^1(t;R_N)/\mu(R_N) = \rho(t;x_o),$$

$$\lim_{N\to\infty} D_N^{1+a}(t;R_N)/\mu(R_N) = \rho(f;x_o)\overline{u}^a(t;x_o); \tag{5.21}$$

$$\lim_{N\to\infty} D_N^5(t;R_N)/\mu(R_N) = e(t;x_o),$$

whenever R_N is any cell selected from the N-th collection, so chosen that the cells close down on the point x_o. We may also assume that the ratio of maximum to minimum diameter of each cell is $\leqq$ some fixed number Q independent of N.

Now, consider the manifolds $\mathcal{M}_N$ in the N-th phase space consisting of <u>all</u> phase particles whose corresponding distributions coincide with the $D_N^\gamma(t_1,R)$ at time t_1 for all cells R of the N-th set. For all of these it is seen that the corresponding $\psi_N^\gamma(t_1;y)$ differ very little from those

of the given D_N^γ. The derivatives with respect to time may differ considerably but those for γ from 1 to 4 are bounded independently of N. It is likely that as $N \to \infty$, the proportion of phase particles on such $\mathfrak{M}_N$ for which the derivatives of the ψ^5 functions fail to be bounded tends to zero. Since these ψ^γ_{Nt} are bounded independently of N and we have uniform convergence, it follows that every phase particle remains on nearby such manifolds for an appreciable length of time, independently of N. On the other hand, the factor σ_N^{-1} in the equations of motion 5.10 suggests that the total speed of a phase particle becomes large with N. This suggests that the total motion of the phase particle is compounded of a rapid motion along a manifold $\mathfrak{M}_N$ and a slow motion into neighboring manifolds. Since there seem to be no other functions besides $\psi^1, \ldots, \psi^5$ which vary slowly with time, it would seem that there are no "invariant submanifolds" of $\mathfrak{M}_N$ so that the projection on $\mathfrak{M}_N$ of the phase particle would come close to every point on $\mathfrak{M}_N$ in a short time interval in the manner stated in the well-known Ergodic Theorem. This is reinforced by the fact that there are N! indistinguishable phase particles obtained from one another by permuting the indices of the particles. The Ergodic Theorem states in such a case that the time average over a sufficiently long time interval, which in our case may tend to zero as $N \to \infty$, would be equal to the space average over $\mathfrak{M}_N$ of any given point function on $\mathfrak{M}_N$. We therefore make the following <u>fundamental assumption</u>.

Assumption 5: The derivatives $\psi^\gamma_t(t_1;y)$ are equal, respectively, to the limits as $N \to \infty$ of the averages over $\mathfrak{M}_N$ of the expressions in terms of the x_j^ρ and u_j^ρ for the derivatives of $\psi^\gamma_{Nt}(t;y)$ given in equations 5.19.

The averages over $\mathfrak{M}_N$ are, of course, to be taken with respect to an appropriate "surface measure" on $\mathfrak{M}_N$. Since the flow in the phase space defined by the equations of motion 5.10 is known to preserve volumes, this measure on $\mathfrak{M}_N$ is "invariant" as is required in the Ergodic Theorem.

5.6 FIRST STEP IN THE AVERAGING PROCESS.

Assumption 5 reduces our problem of finding time averages to that of finding the space averages of certain functions over certain manifolds, a type of problem which is rather standard in Statistical Mechanics. Since this is so and since the formulas are complicated, we shall not carry out this work in all detail.

The expressions in equations 5.19 are seen to be symmetric in the indices. Since each $\mathcal{M}_N$ has the same property we may replace the 1/N times each single sum by one term with $j = 1$ and the sums involving $k \neq j$ by N-1 times the single term with $k = 2$. Thus the averages over $\mathcal{M}_N$ of the expressions for ψ^{γ}_{Nt} are equal to those below

$$iy^{\beta}\exp\left[i(y\cdot x_1)\right]\left\{Mu_1^{\alpha}u_1^{\beta}+(N-1)E\left[iy\cdot(x_2-x_1)\right]v_{12}^{\alpha\beta}/2\right\},$$
$$\text{for } \gamma = 1+\alpha$$

$$iy^{\beta}\exp\left[i(y\cdot x_1)\right]\left\{u_1^{\beta}\bar{e}_1+(N-1)u_1^{\alpha}E\left[iy\cdot(x_2-x_1)\right]v_{12}^{\alpha\beta}/2\right\},\qquad (5.22)$$
$$\text{for } \gamma = 5$$

where E(z) and $v_{12}^{\alpha\beta}$ are defined in equations 5.20, and

$$\bar{e}_1 = (M/2)\left[|u_1|^2+K(N-1)\ \Phi(r_{12}/\sigma_N)\right].\qquad (5.23)$$

Since we already know that the equations for ψ^1_t holds in the limit, we have omitted it here. Each of the quantities in equations 5.22 and 5.23 is a function of (x_1,x_2,u_1) for each fixed y. In order to average a function $f(x_1,x_2,u_1)$ over a manifold $\mathcal{M}$, we first find

$$\lambda(x_1,x_2,u_1) = \left[\mu(\mathcal{M})\right]^{-1}\mu\left[\mathcal{M}(x_1,x_2,u_1)\right]\qquad (5.24)$$

in which μ is the surface measure and $\mathcal{M}(x_1,x_2,u_1)$ denotes the section of $\mathcal{M}$ for which x_1, x_2, and u_1 have their given values. The average is then given by

$$\int_G f(x_1,x_2,u_1)\ \lambda(x_1,x_2,u_1)dx_1dx_2du_1\qquad (5.25)$$

G being the projection of $\mathcal{M}$ on the (x_1, x_2, u_1) space.

For simplicity let us hold N fixed for the moment and denote $\mathcal{M}_N$ by $\mathcal{M}$ and the N-th set of cells by $R_1, \cdots, R_P$. From our definition of $\mathcal{M}$ and of the various distributions, we see that $\mathcal{M}$ consists of all (x_j, u_j) in the phase space such that

(i) there are N_p particles x_j in R_p where

$$N_p = ND_N^1(t_1;R_p)/M, \quad p = 1, \cdots, P$$

(ii) $\sum_j u_j^a = N_p \overline{u}_j^a = ND_N^{1+a}(t_1;R_p)/M$

$$\text{(iii)} \quad \sum_j |u_j|^2 + W_p(x) = 2N_pE_p = 2ND_N^5(t_1;R_p)/M, \text{ where} \tag{5.26}$$

$$W_p(x) = K \sum_j \sum_{\substack{k=1 \\ k\neq j}}^{N} \Phi(r_{jk}/\sigma_N),$$

$$E_p = D_N^5(t_1;R_p)/D_N^1(t_1;R_p)$$

and the sums on j are over those j for which x_j is in R_p.

We wish to reduce the problem of finding the function $\lambda(x_1, x_2, u_1)$ of equation 5.24 to simpler terms. We see from the previous paragraph that $\mathcal{M}$ breaks up into a number of symmetrically placed manifolds $\mathcal{M}_J$ where J stands for a permutation

$$j_{1,1}, \cdots, j_{1,N_1}; j_{21}, \cdots, j_{2,N_2}; \cdots; j_{P,1}, \cdots, j_{P,N_P},$$

and $j_{p,1}, \cdots, j_{p,N_p}$ are those j for which x_j lies in R_p, $p = 1, \cdots, P$. All the manifolds obtained by permuting the $j_{p,}$'s among themselves for each p are identical so that the $\mathcal{M}_J$ are distinct only when the <u>sets</u> $j_{p,k}$ are not all identical. The number of ways in which these P sets of $N_1, \cdots, N_P$ objects can be selected is well known to be

$$\frac{N!}{N_1!\cdots N_p!} \quad (N = N_1+\cdots+N_p).$$

Now, suppose that x_1, is in R_p and x_2 is in R_q. Then $\mathcal{M}_J(x_1,x_2,u_1)$ is empty unless J is such that 1 occurs in the p-th set and 2 in the q-th set. The number of J for which this is the case is

$$\frac{(N-2)!}{N_1!\cdots(N_p-2)!\cdots N_p!} \text{ if } p = q, \text{ and } \frac{(N-2)!}{N_1!\cdots(N_p-1)!\cdots(N_q-1)!\cdots N_p!}$$

if $p \neq q$.

Hence if x_1 is in R_p and x_2 is in R_2, we have

$$\lambda(x_1,x_2,u_1) = \begin{cases} N_p(N_{p-1})\kappa_{pp}(x_1,x_2,u_1)/N(N-1) \text{ if } p = q, \\ N_pN_q\kappa_{pq}(x_1,x_2,u_1)/N(N-1) \text{ if } p \neq q, \text{ where} \end{cases} \tag{5.27}$$

$$\kappa_{pq}(x_1,x_2,u_1) = \left[\mu(\mathcal{M}_J)\right]^{-1}\mu\left[\mathcal{M}_J(x_1,x_2,u_1)\right]$$

for a fixed J for which 1 is in the p-th set and 2 is in the q-th.

5.7 DETERMINATION OF THE FUNCTIONS κ_{pq}

In this section, we sketch briefly the determination of the functions κ_{pq}. There are really only two distinct cases: $p = q$ and $p \neq q$. Since the results must come out in terms of the constants N, N_p, $\overline{u}_p^{\alpha}$, and E_p, we may take $p = q = 1$ in the first case and $p = 1$, $q = 2$ in the second; the results for the general p and q may then be read off.

In order to avoid complicated notation involving the $j_{p,k}$, we introduce a double subscript notation for the x's and u's in which $x_{p1},\cdots,x_{p,N_p}$ are the x's in R_p for each p. Since we wish to exhibit the dependence on x_1,x_2, and u_1 we assume the alternative notations

$$x_1 = x_{1,N_1},\ u_1 = u_{1,N_1} \text{ in both cases}$$

$$x_2 = x_{1,N_1-1} \text{ if } p = q = 1, \text{ and } x_2 = x_{2,N_2} \text{ if } p = 1,\ q = 2, \tag{5.28}$$

and shall use these notations interchangeably. Our given manifold $\mathcal{M}_J$ can then be described by

$$x_{pj} \text{ in } R_p \text{ for } j = 1,\cdots,N_p \text{ and } p = 1,\cdots,P;$$

$$\sum_{j=1}^{N_p}(u^\alpha_{pj}-\bar{u}^\alpha_p) = 0, \quad \alpha = 1,2,3;\ p = 1,\cdots,P;$$

$$\sum_{j=1}^{N_p}\left|u_{pj}-\bar{u}_p\right|^2 + W_p(x) = N_p(2E_p-\left|\bar{u}_p\right|^2) = 2N_pE^*_p \tag{5.29}$$

where

$$W_p(x) = K \sum_{\substack{j,k=1\\ j\neq k}}^{N_p} \Phi(\left|x_{pj}-x_{pk}\right|/\sigma_N)+K \sum_{\substack{q=1\\ q\neq p}}^{P} \sum_{j=1}^{N_p} \sum_{k=1}^{N_p} \Phi(\left|x_{pj}-x_{qk}\right|/\sigma_N)$$

$$E^*_p = E_p - \left|\bar{u}_p\right|^2/2. \tag{5.30}$$

The derivation of the last equation makes use of the fact that

$$\sum_{j=1}^{N_p}\left|u_{p,j}-\bar{u}_p\right|^2 = \sum_{j=1}^{N_p}\left|u_{p,j}\right|^2 - N_p\left|\bar{u}_p\right|^2 ;$$

this follows immediately from the first equations.

Since all we want is the surface area of $\mathcal{M}_J$ and $\mathcal{M}_J(x_1,x_2,u_1)$, we may introduce new variables $v_{p,j}$ defined by

$$v^\alpha_{p,j} = u^\alpha_{p,j} - \bar{u}^\alpha_p. \tag{5.31}$$

The equations 5.29 can then easily be solved for v^1_{p1}, v^2_{p1}, v^3_{p1}, and v^1_{p2} in terms of the other $v^\alpha_{p,j}$ and all the

$x^{\beta}_{q,k}$. For each p, we obtain 2 soultions so $\mathfrak{M}_J$ and $\mathfrak{M}_J(x_1,x_2,u_1)$ break up into 2^p parts of equal area on each of which the solutions above are single-valued; we have chosen our variables so that x_1,x_2, and u_1 are among the independent variables. The element of surface area on $\mathfrak{M}_J$ and $\mathfrak{M}_J(x_1,x_2,u_1)$ can be found by standard formulas; one can then find the area of $\mathfrak{M}_J(x_1,x_2,u_1)$ by integrating the area element with respect to all the independent variables except x_1,x_2, and u_1 and can then find that of $\mathfrak{M}_J$ by integrating that result with respect to (x_1,x_2,u_1).

More specifically, suppose that G is the projection of $\mathfrak{M}_J$ on the (x_1,x_2,u_1) space and, for each set (x_1,x_2,u_1) in G, suppose $G(x_1,x_2,u_1)$ is the projection of $\mathfrak{M}_J(x_1,x_2,u_3)$ on the space of the remaining x_{pj}. Having chosen x_1,x_2, and u_1 in G and a set of remaining x_{pj} in $G(x_1,x_2,u_1)$, it turns out from the fact that the solutions for v^{α}_{p1} and v^{1}_{p2} depend only on the v^{α}_{pj} of the p-th set that the area element is a product of functions of the v_{pj} only and that the domains of integration of the v_{pj} are independent. Thus the integration with respect to the v's breaks up into a product of integrals of the form

$$\int_{S_p} f_p(x_1,x_2,u_1,\hat{x}_{12};v_{pj})dv_{pj}$$

where the S_p are ellipsoids whose positions and dimension depend on $(x_1,x_2,u_1,\hat{x}_{12})$ and the f_p are simple functions of the v_{pj}; here $\hat{x}_{12}$ denotes all the x_{pj} except x_1 and x_2. The result is

$$\mu\left[\mathfrak{M}_J(x_1,x_2,u_1)\right] = \int_{\Gamma(x_1,x_2,u_1)} g_1(x_1,x_2,u_1,\hat{x}_{12})\cdots g_p(x_1,x_2,u_1,\hat{x}_{12})d\hat{x}_{12}, \tag{5.32}$$

where

$$g_1 = K_1\left[1-V_1(x)/E_1^* - |u_1-\bar{u}_1|^2/2E_1^*(N_1-1)\right]^{(3N_1-8)/2}$$

$$g_p = K_p\left[1-V_p(x)/E_p^*\right]^{(3N_p-5)/2}, \quad p > 1, \tag{5.33}$$

$$K_1 = 2^{1/2}(N_1-1)^{-3/2}\pi^{3(N_1-2)/2}(2N_1E_1^*)^{(3N_1-8)/2} \Big/ \Gamma\left[3(N_1-2)/2\right]$$

$$K_p = 2^{1/2}N_p^{-3/2}\pi^{3(N_p-1)/2}(2N_pE_p^*)^{(3N_p-5)/2} \Big/ \Gamma\left[3(N_p-1)/2\right], \quad p>1$$

$$V_p(x) = W_p(x)/2N_p, \quad p = 1,\cdots,P,$$

and $\Gamma(z)$ is the gamma function. Since we are ultimately letting N (and hence each N_p by equations 5.21) $\rightarrow \infty$, we replace g_1 by its very accurate asymptotic formula

$$g_1 = K_1\left[1-V_1(x)/E_1^*\right]^{(3N_1-8)/2} \exp\left[-\alpha(x)|u_1-\bar{u}_1|^2\right], \text{ where}$$

$$\alpha(x) = 3/4\left[E_1^* - V_1(x)\right] \tag{5.34}$$

We now wish to exhibit the behavior of the integral in equation 5.32 as a function of (x_1,x_2). We assume first that J is such that x_1 and x_2 are both on R_1. By referring to the definition of $W_p(x)$ in equation 5.30 and of $V_p(x)$ in equation 5.33, we may write

$$V_1(x) = K\,\Phi(r_{12}/\sigma_N)/N_1 + \bar{V}_1(x) \tag{5.35}$$

where $\bar{V}_1(x)$ contains all the remaining terms. We write

$$\left[1-V_1(x)/E_1^*\right] = \left[1-K\,\Phi(r_{12}/\sigma_N)/N_1E_1^*\right]\cdot\left[1-\bar{V}_1(x)/E_1^{**}\right], \text{ where}$$

$$E_1^{**} = E_1^* - K\,\Phi(r_{12}/\sigma_N)/N_1 \tag{5.36}$$

Using the standard asymptotic formula in the preceding paragraph, see that we may write

$$g_1 = K_1 \cdot \exp\left[-3K\,\Phi(r_{12}/\sigma_N)/2E_1^*\right] \cdot \left[1-\bar{V}_1(x)/E_1^{**}\right]^{(3N_1-8)/2} \cdot$$

$$\cdot \exp\left[-a(x)\,|u_1-\bar{u}_1|^2\right] .$$

Since the first exponential factor does not depend on the variables of integration in equation 5.32, we may take it out in front of the integral. In a similar way, if x_1 is in R_1 and x_2 is in R_2, we find that we may define $\bar{V}_1(x)$ and $\bar{V}_2(x)$ properly to obtain

$$g_1 = K_1 \exp\left[-3K\,\Phi(r_{12}/\sigma_N)/4E_1^*\right] \cdot \left[1-\bar{V}_1(x)/E_1^{**}\right]^{(3N_1-8)/2} \cdot$$

$$\cdot \exp\left[-a(x)\,|u_1-\bar{u}_1|^2\right]$$

$$g_2 = K_2 \exp\left[-3K\,\Phi(r_{12}/\sigma_N)/4E_2^*\right] \cdot \left[1-\bar{V}_2(x)/E_2^{**}\right]^{(3N_1-5)/2}$$

and the first exponential factors may be taken out in front of the integral.

We wish to exhibit the dependence of $\mu\left[m_J(x_1,x_2,u_1)\right]$ on u_1. We can think of the integral in equation 5.32 as equal to an average value of the exponential involving u_1 times the integral of the remaining factors. We need to discuss this average value. Since we have assumed that σ_N is small in comparison with the dimensions of R_p, we note that the sums

$$K \sum_{\substack{q=1\\ q\neq p}}^{p} \sum_{k=1}^{N_q} \Phi(|x_{pj}-x_{qk}|/\sigma_N)$$

are all small in comparison with

$$K \sum_{\substack{k=1\\ k\neq j}}^{N_p} \Phi(|x_{p,j}-x_{p,k}|/\sigma_N)$$

except for j such that x_{pj} is within a distance comparable with σ_N of the boundary of R_p provided that $\Phi(\rho)$ tends to zero as $\rho \rightarrow \infty$ rapidly enough. Thus it would seem that we might replace the functions $W_p(x)$ in equations 5.30 (and hence the corresponding V_p and $\bar{V}_p$) by the first sums without making much error in the desired average. We formalize this in the following assumption:

Assumption 6: In calculating the asymptotic value as $N \rightarrow \infty$ of the average value of the factor $\exp\left[-\alpha(x)\cdot|u_1-\bar{u}_1|^2\right]$ in the integral in equation 5.32 for $\mu\left[W_J(x_1,x_2,u_1)\right]$, it is possible for each N to replace the functions $V_p(x)$ $\left[\text{and } \bar{V}_1(x) \text{ and } \bar{V}_2(x)\right]$ by the functions $V_p^*(x)$ obtained by omitting all terms of the form $\Phi(|x_{p,j}-x_{q,k}|/\sigma_N)$ for $q \neq p$ and simultaneously to choose proper independent domains of integration $G_p(x_1,x_2,u_1)$ for the x_{pj}.

When this is done the integral breaks up into a product of integrals I_p in which $I_3,\cdots,I_P$ are independent of (x_1,x_2,u_1) and I_2 is independent of u_1. In the case where x_1 and x_2 are both in R_1, I_2 is also independent of (x_1,x_2) and have

$$I_1 = K_1\exp\left[-3K\,\Phi(r_{12}/\sigma_N)/2E_1^*\right]\int_{G_1}\left[\exp -\alpha^*(x)\,|u_1-u_1|^2\right] \tag{5.37}$$

$$\left[1-V_1^*(x)/E_1^{**}\right]^{(3N_1-8)/2} dx_{1,1}\cdots dx_{1,N,-2}$$

in which E_1^{**} is defined in equation 5.36 and

$$V_1^*(x) = \frac{A}{N_1}\sum_{j=1}^{N_1-2}\left[\Phi(|x_{1,j}-x_1|/\sigma_N)+\Phi(|x_{1j}-x_2|/\sigma_N)\right]$$

$$+\frac{K}{N_1}\sum_{\substack{j,k=1\\ j\neq k}}^{N_1-2}\Phi(|x_{1j}-x_{1k}|/\sigma_N).$$

$$\alpha^*(x) = 3/4\left[E_1^{**}-V_1^*(x)\right]$$

To simplify this further, we assume

Assumption 7: In computing an asymptotic formula for the average of the exponential factor in the integral I_1 of equation 5.37, we may replace $V_1^{*}(x)$ by the function

$$V_1^{**}(x) = \frac{K}{N_1-2} \sum_{\substack{j,k=1 \\ j\neq k}}^{N_1-2} \Phi\left(|x_{1j}-x_{1k}|/\sigma_N\right), \qquad (5.38)$$

simultaneously enlarging the domain of integration G_1 to include all $x_{11},\cdots x_{1,N_1-2}$ for which

$$\beta \leqq V_1^{**}(x) \leqq E_1^{**} = E_1^{*} - K\,\Phi(r_{12}/\sigma_N)/N_1.$$

β being the minimum of V_1^{**} for the given values of σ_N and N_1.

When this substitution is made, the integral I_1 depends on (x_1,x_2) only through the value of E_1^{**} which tends to E_1^{*}. To investigate this integral, let $\mu_{N_1}(\lambda)$ be the measure of the manifold

$$V_1^{**}(x_{11},\cdots,x_{1,N_1-2}) = \lambda\,, \quad \beta \leqq \lambda \leqq E_1^{**}\,,$$

and let us denote E_1^{**} by h. Then the integral in I_1 becomes

$$\int_{\beta}^{h} \exp\left[-3|u_1-\bar{u}_1|^2/4(h-\lambda)\right]\cdot(1-\lambda/h)^{(3N_1-8)/2}\mu_{N_1}(\lambda)d\lambda\,. \qquad (5.39)$$

The high power of $(1-\lambda/h)$ occurring in the integrand suggest strongly that an asymptotic formula for the average of the exponential factor in this integral would be obtained by setting $\lambda = \beta$. On the other hand, as $N_1 \longrightarrow \infty$, the functions $\mu_{N_1}(\lambda)$ may tend rapidly to zero for small values of λ. What happens awaits a further study of these functions μ_{N_1}. However, in order to obtain a definite result we make the assumption:

Assumption 8: An asymptotic value for the average value of the exponential factor in the integral in equation 5.39 is obtained by setting $\lambda = \beta$.

If this assumption is not made, the average value would occur for some other intermediate value $\bar{\lambda}$ which might depend on u_1 as well as on β and h. In this case the average would not be exactly an exponential function but would behave somewhat like one and, at any rate would depend only on β, h, and $|u_1-\bar{u}_1|^2$.

Acceptance of this assumption focuses attention on β. From the form of $V_1^{**}(x)$ (equation 5.38), it follows that if the size of R_1 is increased and σ_N is simultaneously increased so that $\sigma_N^3/\mu(R_1)$ is kept constant and if N_1 is fixed, then β is unchanged. Also, if the shape of R_1 is held in bounds as described in the original selection of the R_p and if $\Phi(\rho) \rightarrow 0$ rapidly enough as $\rho \rightarrow \infty$, it is practically evident that β will depend essentially only on the combination

$$\rho_1 = N_1 \sigma_N^3/\mu(R_1) = M\cdot(N_1/N)/m(R_1) = D_N^1(t_1;R_1)/\mu(R_1) \quad (5.40)$$

But we have seen in equation 5.21 that

$$\lim_{N\to\infty} \rho_1 = \lim_{N\to\infty} D_N^1(t_1;R_1)/\mu(R_1) = \rho(t_1;x_0)$$

if the R_1 are selected to close down on the point x_0. Hence we make the assumption:

Assumption 9: Asymptotically

$$\beta = \beta(\rho_1) \ \underline{\text{where}} \ \rho_1 = D_N^1(t_1;R_1)/\mu(R_1).$$

Using assumptions 6 through 9, we find that if x_1 and x_2 are both on R_1,

$$\mu\left[\mathfrak{M}_J(x_1,x_2,u_1)\right] = K_1\cdots K_p\cdot I_2\cdots I_p\cdot I\cdot$$
$$\cdot\exp\left[-B(\rho_1,E_1^*)\,|u_1-\bar{u}_1|^2 - 3K\,\Phi(r_{12}/\sigma_N)/2E_1^*\right] \quad (5.41)$$

where

$$B(\rho_1, E_1^*) = 3/4\left[E_1^* - \beta(\rho_1)\right] ,$$

$$I = \int_{\beta(\rho_1)}^{E_1^*} (1-\lambda/E_1^*)^{(3N_1-8)/2} \mu_{N_1}(\lambda)d\lambda. \tag{5.42}$$

Since $\mu(\mathcal{M}_J)$ is the integral of $\mu\left[\mathcal{M}_J(x_1,x_2,u_1)\right]$, we see that the function π_{11} defined in equation 5.27 must be some constant times the exponential function in equation 5.41, the constant chosen so the integral of π_{11} with respect to (x_1,x_2,u_1) is 1. But the form of this function shows that we may extend the original projection G of $\mathcal{M}_J$ on the (x_1,x_2,u_1) space to include all u_1 and all x_1 and x_2 on R_1 (previously x_1 and x_2 had to remain at some positive distance apart $\rightarrow 0$ as $N \rightarrow \infty$ and $|u_1-\bar{u}_1|$ had to remain less than some large number $\rightarrow \infty$ as $N \rightarrow \infty$) without affecting the asymptotic formula for π_{11}. Supposing

$$\pi_{11} = A \exp\left[-B|u_1-\bar{u}_1|^2 - 3K\,\Phi(r_{12}/\sigma_N)/2E_1^*\right]$$

and the integral of π_{11} with respect to (x_1,x_2,u_1) is 1, we obtain

$$1 = A\int_{R_1} dx_1 \int_{R_1} \exp\left[-3K\,\Phi(r_{12}/\sigma_N)/2E_{12}^*\right]dx_2 \int_{-\infty}^{\infty} \exp\left[-B|u_1-\bar{u}_1|^2\right]du_1.$$

Since, by a well known formula

$$\int_{-\infty}^{\infty} \exp\left[-B|u_1-\bar{u}_1|^2\right]du_1 = (\pi/B)^{3/2} ,$$

and since the exponential involving Φ is practically 1 except when r_{12} is of the order of σ_N and since $\sigma_N^3/\mu(R_1) \rightarrow 0$ we see that we may write, asymptotically,

$$\pi_{11}(x_1,x_2,u_1) =$$

$$\left[B(\rho_1,E_1^*)/\pi\right]^{3/2} \left[\mu(R_1)\right]^{-2} \exp\left[-B_1|u_1-\bar{u}_1|^2 - 3K\,\Phi(r_{12}/\sigma_N)/2E_1^*\right].$$

An entirely similar argument shows that

$$\pi_{12} = \left[B(\rho_1, E_1^*)/\pi\right]^{3/2} \left[\mu(R_1)\right]^{-1} \left[\mu(R_2)\right]^{-1} \cdot$$
$$\cdot \exp\left[-B_1 |u_1 - \bar{u}_1|^2 - 3K\, \Phi(r_{12}/\sigma_N)(E_1^{*-1} + E_2^{*-1})/4\right].$$

We may therefore read off our general results:

$$\pi_{pp} = (B_p/\pi)^{3/2}\left[\mu(R_p)\right]^{-2} \exp\left[-B_p |u_1 - \bar{u}_p|^2 - 3K\, \Phi(r_{12}/\sigma_N)/2E_p^*\right] \tag{5.43}$$

$$\pi_{pq} = (B_p/\pi)^{3/2}\left[\mu(R_p)\right]^{-1} \left[\mu(R_q)\right]^{-1} \cdot$$
$$\cdot \exp\left[-B_p |u_1 - \bar{u}_p|^2 - 3K\, \Phi(r_{12}/\sigma)(E_p^{*-1} + E_q^{*-1})/4\right]$$

5.8 THE LIMITING EQUATIONS.

In this section we apply the results of the preceding sections to determine the quantities $\Psi_t^{\gamma}(t;y)$ and will also determine the form of the function $\varphi(t;y,v) = X_t(t;y,v)$ of § 5.4. These, in turn, will lead to our proposed equations of motion in the $(t;x)$ space and to the form of the limiting distribution of coordinates and velocities.

We first determine $\varphi(t;y,v) = X_t(t;y,v)$ which will be determined, using the fundamental assumption 5 and the symmetry of $\mathcal{M}$ as the limit of the average over $\mathcal{M}$ of

$$M \exp\left[i(y \cdot x_1 + v \cdot u_1)\right]$$

According to equations 5.25, 5.27, and 5.43, this average will be given by the limit of

$$\int \lambda(x_1, x_2, u_1) \exp\left[i(y \cdot x_1 + v \cdot u_1)\right] dx_1 dx_2 du_1$$

$$= \sum_{p=1}^{P} \rho_p \cdot (N_p/N) \left[\mu(R_p)\right]^{-1} \int_{R_p} \exp\left[i(y \cdot x_1)\right] dx_1 \cdot$$

$$\cdot \int_{R_p} \exp\left[-3\,\Phi(r_{12}/\sigma_N)/2E_p^*\right] dx_2 \cdot$$

$$\cdot \int_{-\infty}^{\infty} \exp\left[i(v\cdot u_1)\right] (B_p/\pi)^{3/2} \exp\left[-B_p\,|u_1-\bar{u}_p|^2\right] du_1$$

$$+ \sum_{p=1}^{P} \sum_{\substack{q=1\\ q\neq p}}^{P} \left(\rho_p \cdot (N_q/N)\left[\mu(R_q)\right]^{-1} \int_{R_p} \exp\left[i(y\cdot x_1)\right] dx_1 \cdot \right.$$

$$\cdot \int_{R_q} \exp\left[-3\,\Phi(E_p^{*-1}+E_q^{*-1})/4\right] dx_2 \cdot$$

$$\cdot \int_{-\infty}^{\infty} \exp\left[i(v\cdot u_1)\right] \; (B_p/\pi)^{3/2} \exp\left[-B_p|u_1-\bar{u}_1|^2\right] du_1$$

where ρ_1 was defined in equation 5.40. Now $N_1+\cdots+N_p = N$, and $\rho_p \rightarrow \rho(t_1;x_o)$ and [see equations 5.30, 5.26, and 5.21]

$$\lim E_p^* = \mathcal{E}(t_1;x_o) = e(t_1;x_o)/\rho - \left|\bar{u}(t_1;x_o)\right|^2/2$$

as R_p closed down on x_o, where $\bar{u}^a(t;x)$ is the mass velocity vector and $\mathcal{E}(t;x)$, defined by this equation, is the <u>specific internal energy per unit mass</u>. In the limit, since the exponential involving Φ is 1 most of the time, we obtain

$$\varphi(t;y,v) = \int_{-\infty}^{\infty} \exp\left[i(y\cdot x_1+v\cdot u_1)\right] \rho(t;x_1)(B/\pi)^{3/2}\cdot$$
$$\cdot \exp\left[-B\left|u_1-\bar{u}(t,x)\right|^2\right] dx_1 du_1 \quad (5.44)$$

where $B = B(\mathcal{E},\rho) = 3/4\left[\mathcal{E}-\beta(\rho)\right]$.

This is seen to be the Fourier transform of the function

$$\pi(t;x,u) = \rho(t;x)\cdot(B/\pi)^{3/2} \exp\left[-B\left|u-\bar{u}(t;x)\right|^2\right] \quad (5.45)$$

which is the density function for the distribution of coordinates and velocities.

We may read off the averages of

$$Mu_1^{a}u_1^{\beta}\exp\left[i(y\cdot x_1)\right] \quad \text{and} \quad Mu_1^{\beta}\left|u_1\right|^2 \exp\left[i(y\cdot x_1)\right]$$

from the result in equation 5.44 by expanding the exponential in powers of the v^{α} and carrying out the integration with respect to u_1 or by carrying through the averaging process directly; the latter process evidently gives

$$\int_{-\infty}^{\infty} \exp\left[i(y\cdot x_1)\right] \rho(t;x_1)dx_1 \int_{-\infty}^{\infty}(B/\pi)^{3/2} u_1^{\alpha}u_1^{\beta}\exp\left[-B\left|u_1-\bar{u}_1\right|^2\right] du_1$$
$$= \int_{-\infty}^{\infty} \exp\left[i(y\cdot x_1)\right] \rho(t;x_1)\left[\bar{u}^{\alpha}(t;x_1)\bar{u}^{\beta}(t;x_1)+\delta^{\alpha\beta}/2B\right]dx_1$$
$$(5.46)$$
$$\int_{-\infty}^{\infty} \exp\left[i(y\cdot x_1)\right] \rho(t;x_1)dx_1 \int_{-\infty}^{\infty}(B/\pi)^{3/2} u_1^{\beta}\left|u_1\right|^2 \exp\left[-B\left|u_1-\bar{u}_1\right|^2\right] du_1$$
$$= \int_{-\infty}^{\infty} \exp\left[i(y\cdot x_1)\right] \rho(t;x_1)\left[\bar{u}_1^{\beta}\left|\bar{u}_1\right|^2+5\bar{u}_1^{\beta}/2B\right]dx_1$$

where $\delta^{\alpha\beta}$ is the usual Kronecker delta defined by

$$\delta^{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha=\beta \\ 0 & \text{if } \alpha\neq\beta. \end{cases}$$

In order to complete the determination of the φ_t^{γ}, we see from equations 5.22 and 5.23 that we need to compute also the averages of

$$KM(N-1)u_1^{\beta}\ \Phi(r_{12}/\sigma_N)\exp\left[i(y\cdot x_1)\right]$$
$$(N-1)E\left[iy\cdot(x_2-x_1)\right]v_{12}^{\alpha\beta}\ \exp\left[i(y\cdot x_1)\right] \qquad (5.47)$$
$$(N-1)u_1^{\alpha}\ E\left[iy\cdot(x_2-x_1)\right]v_{12}^{\alpha\beta}\ \exp\left[i(y\cdot x_1)\right]$$

We note that, in carrying the averaging process in equations for these quantities, we may first carry out the integrations with respect to u_1. In the first and third quantities in equations 5.47, this results merely in the constant factors $\bar{u}_p^{\beta}$ and $\bar{u}_p^{\alpha}$ which can be taken entirely outside the integrals along with ρ_p; in the second the result of the integration is just 1. Also the factor $\exp\left[i(y\cdot x_1)\right]$ is the same for all terms in the sum. If we

denote the averages of the respective quantities in equations 5.47 by Q_1^{β}, $Q_2^{\alpha\beta}$, and Q_3^{β}, we therefore obtain

$$Q_1^{\beta} = \sum_{p=1}^{P} \rho_p \bar{u}_p^{\beta} \int_{R_p} \exp\left[i(y \cdot x_1)\right] A(x_1) dx_1$$

$$Q_2^{\alpha\beta} = \sum_{p=1}^{P} \rho_p \int_{R_p} \exp\left[i(y \cdot x_1)\right] C^{\alpha\beta}(y, x_1) dx_1$$

$$Q_3^{\beta} = \sum_{p=1}^{P} \rho_p \bar{u}_p^{\alpha} \int_{R_p} \exp\left[i(y \cdot x_1)\right] C^{\alpha\beta}(y, x_1) dx_1$$

where

$$A(x_1) = \sum_{q=1}^{P} (\rho_q/M) \int_{R_q} K(N-1)\, \Phi(r_{12}/\sigma_N) \cdot$$
$$\cdot \exp\left[-3K\, \Phi(r_{12}/\sigma_N)/4E_p^{*} - 3K\, \Phi(r_{12}/\sigma_N)/4E_q^{*}\right] dx_2$$

$$C^{\alpha\beta}(y; x_1) = \sum_{q=1}^{P} (\rho_q/M) \int_{R_q} K(N-1) \cdot$$
$$\cdot \exp\left[-3K\, \Phi(r_{12}/\sigma_N)/4E_p^{*} - 3K\, \Phi(r_{12}/\sigma_N)/4E_q^{*}\right] \cdot$$
$$\cdot E\left[iy \cdot (x_2 - x_1)\right] \left[-(r_{12}/\sigma_N) \cdot \Phi'(r_{12}/\sigma_N) \cdot (x_1^{\alpha} - x_2^{\alpha})(x_1^{\beta} - x_2^{\beta})/r_{12}^{2}\right] \cdot$$
$$\cdot\, dx_2$$

In order to evaluate these integrals, we set $\xi = (x_2 - x_1)/\sigma_N$. Then

$$dx_2 = \sigma_N^3 d\xi \quad \text{and} \quad (N-1)\, \sigma_N^3 = M(1-N^{-1})/D \qquad (5.48)$$

and the integrations are now extended over new cells R'_q each obtained from the original R_q by first translating it through the vector $-x_1$ and then magnifying the result in the ratio σ_N^{-1}. Of these new cells, only R'_p contains the origin and the others are all far from the origin

unless x_1 is within a distance comparable to σ_N from the boundary of R_p. Hence, if x_1 is confined to the part of R_p at a distance d from the boundary which remains as $N \to \infty$ comparable with the dimensions of R_p, we see that all the R'_q go off to infinity and the integrals over them contribute nothing to the integral in the limit while R'_p expands to include the whole ξ -space. Since $|E(iz)| \leq 1$ for all real z and $E(0) = 1$, we see using equation 5.48 that the limiting values of $A(x_1)$ and $C^{\alpha\beta}(y;x_1)$ are

$$A(x_1) = \frac{\rho_p}{D}\int_{-\infty}^{\infty} K\,\Phi(|\xi|)\exp\left[-3K\,\Phi(|\xi|)/2E_p^*\right]d\xi =$$

$$4\pi K\int_0^{\infty} w^2\,\Phi(w)\exp\left[-3K\,\Phi(w)/2E_p^*\right]dw = KA(E_p^*/K)\ \rho_p/D$$

$$C^{\alpha\beta}(y,x\ ) = \frac{K\rho_p}{D}\ (\ \delta^{\alpha\beta}/3)\cdot 4\pi\int_0^{\infty} -w^3\,\Phi'(w)\exp\left[-3K\ \Phi(w)/2E_p^*\right]dw =$$

$$\delta^{\alpha\beta}KC(E_p^*/K)\ \rho_p/D$$

where $A(s)$ and $C(s)$ are dimensionless functions of s only explicitly defined by these equations in terms of s and the potential function Φ . Therefore, the limiting values of the Q's are

$$Q_1^{\ \beta}(y) = (K/D)\int_{-\infty}^{\infty}\rho^2\bar{u}^{\beta}A(\ \varepsilon/K)\exp\left[i(y\cdot x)\right]dx$$

$$Q_2^{\alpha\beta}(y) = (K/D)\ \delta^{\alpha\beta}\int_{-\infty}^{\infty}\rho^2 C(\ \varepsilon/K)\exp\left[i(y\cdot x)\right]dx$$

$$Q_3^{\ \beta}(y) = (K/D)\int_{-\infty}^{\infty}\rho^2\bar{u}^{\beta}C(\ \varepsilon/K)\exp\left[i(y\cdot x)\right]dx$$

Inserting these results in equations 5.22 and adding in the equation for $\psi'_t(t;y)$, we obtain

$$\psi^1_t(t;y) = iy^{\alpha}\,\psi^{1+\alpha}(t;y) \qquad (5.49)$$

$$\psi^{1+\alpha}_t(t;y) = iy^{\beta}\int_{-\infty}^{\infty}\rho\bar{u}^{\alpha}\bar{u}^{\beta}\exp\left[i(y\cdot x)\right]dx +$$

$$+ (iy^{\alpha}/2)\int_{-\infty}^{\infty}\rho\left[B^{-1}+K\rho C(\varepsilon/K)/D\right]\exp\left[i(y\cdot x)\right]dx$$

$$\Psi_t^5(t;y) = (iy^{\beta}/2)\int_{-\infty}^{\infty}\rho\bar{u}^{\beta}\left[|u|^2+5/2B+K\rho A(\varepsilon/K)/D+K\rho C(\varepsilon/K)/D\right]$$
$$\cdot\exp\left[i(y\cdot x)\right]dx$$

Since $\psi^1,\cdots,\psi^5$ are the Fourier transforms of

$$\rho,\ \rho\bar{u}^1,\ \rho\bar{u}^2,\ \rho\bar{u}^3,\ \text{and } e(t;x) = \rho\left[\varepsilon(t;x)+|u|^2/2\right],$$

we see that the equations 5.49 are just the Fourier transforms of the equations

$$\rho_t + (\rho\bar{u}^{\alpha})_{x^{\alpha}} = 0$$

$$(\rho\bar{u}^{\alpha})_t + (\rho\bar{u}^{\alpha}\bar{u}^{\beta})_{x^{\beta}} + p_{x^{\alpha}} = 0 \qquad (5.50)$$

$$\frac{\partial}{\partial t}\left[\rho(\varepsilon+|\bar{u}|^2/2)\right]+$$

$$+\frac{1}{2}\frac{\partial}{\partial x^{\beta}}\left\{\rho\bar{u}^{\beta}\left[|u|^2+5/2B+K\rho A(\varepsilon/K)/D+K\rho C(\varepsilon/K)/D\right]\right\}=0$$

where we have defined the **pressure** p by

$$p = (\rho/2)\left[B^{-1}+K\rho C(\varepsilon/K)/D\right] = (\rho/2)\left\{4\left[\varepsilon-\beta(\rho)\right]/3+K\rho C(\varepsilon/\rho)/D\right\}$$
$$(5.51)$$

By using the first equation in 5.50 to simplify the second, using the first and second to simplify the third, and introducing the **heat flux vector** q^{α} defined by

$$q^{\alpha} = \frac{1}{2}\rho\bar{u}^{\alpha}\left[5/2B+K\rho A(\varepsilon/\rho)/D+K\rho C(\varepsilon/\rho)/D-2\varepsilon\right]-p\bar{u}^{\alpha}, \qquad (5.52)$$

we obtain the standard equations

$$\rho_t + (\rho \bar{u}^{\alpha})_{x^{\alpha}} = 0$$

$$\bar{u}^{\alpha}_t + \bar{u}^{\beta} \bar{u}^{\alpha}_{x^{\beta}} + p_{x^{\alpha}} = 0 \qquad (5.53)$$

$$\varepsilon_t + \bar{u}^{\alpha} \varepsilon_{x^{\alpha}} + \rho^{-1} \left[q^{\alpha}_{x^{\alpha}} + p u^{\alpha}_{x^{\alpha}} \right] = 0$$

CHAPTER 6

ON THE APPLICATION OF DIMENSIONAL ANALYSIS TO UNDERGROUND EXPLOSIONS

6.1 INTRODUCTION

Dimensional analysis treats the general forms of equations that describe natural phenomena. It arises from an attempt to apply the concepts of geometrical similarity, ratio and proportion to a physical problem. In the following we are specifically concerned with the application of dimensional analysis, in contrast to dimensional reasoning, to the problem of analysing the movement of earth waves due to underground explosions.

Dimensional reasoning is by no means new in this field. Model laws derived in this way were apparently first proposed by C. W. Lampson [7] and since then have been used by other investigators. However their mode of derivation leaves something to be desired from an over-all point of view. It is felt that a more general discussion of the principles of dimensional analysis and their application would be of help to others faced with similar problems.

In the following the assumptions underlying the theory of dimensional analysis are reviewed and the fundamental Pi theorem is stated. The method of computing the unit-free relations is explained and application is made to the problem of determining the most general dimensionless function forms for underground explosions. From these, the model laws used by previous investigators, easily follow.

6.2 THE Pi THEOREM

The assumptions underlying the theory of dimensional analysis have been summarized by G. Birkhoff [1] They are: (I) There are certain independent "fundamental units" q_i such that for any positive real number $a_i (i = 1 \cdots n)$ we can "change units" according to the formula

$$T(q_i) = \alpha_i q_i \quad (\alpha_i > 0) \tag{6.1}$$

(In the following q_i are length, time and mass). II) There are "derived quantities" Q_j (such as density say) which are homogeneous in the sense that under equation 6.1 each Q_j is multiplied by a "conversion factor" given by

$$T(Q_j) = Q_j \alpha_1^{a_{j1}} \cdots \alpha_n^{a_{jn}} \tag{6.2}$$

The exponents a_{jk} are called the "dimensions" of Q_j. If they are all zero, then Q_j is called dimensionless. III) The quantity Q_1 is determined by $Q_2 \cdots Q_r$ through a relation

$$Q_1 = f(Q_2 \cdots Q_r) \tag{6.3}$$

IV) Equation 6.3 is unit free in the sense of being preserved by any transformation of equation 6.1. V) The quantities $Q_1 \cdots Q_r$ involve all $\underline{n}$ fundamental units. With these assumptions the Pi theorem of Vaschy and Buckingham may be formulated as follows.

Theorem 6.1. Let the positive variables $Q_1 \cdots Q_r$ transform by equation 6.2 under all changes of equation 6.1 in the fundamental units $q_1 \cdots q_n$. Let $m \leq n$ be the rank of the matrix $\|a_{jk}\|$ defined by equation 6.2. Then the assertion that

$$g(Q_1 \cdots Q_r) = 0 \tag{6.4}$$

is a unit-free relation, is equivalent to a condition of the form

$$\varphi(\Pi_1 \cdots \Pi_{r-m}) = 0 \tag{6.5}$$

for suitable dimensionless power products $\Pi_1 \cdots \Pi_{r-m}$ of the Q_i.

The proof of the theorem, including a critical discussion of the assumptions can be found in [1].

6.3 SYSTEMATIC DETERMINATION OF THE Π's

The systematic calculation of a complete set of dimensionless products may proceed as follows. Consider

the "derived quantities" $Q_j (j = 1 \cdots r)$ which are derived, for example, from the three "fundamental units" q_i i.e., the length $[L]$, the time $[T]$ and the mass $[M]$. The dimensions of the quantities Q_j can be written as

$$\begin{aligned} &Q_1 \left[L^{a_{11}} M^{a_{12}} T^{a_{13}} \right] \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &Q_r \left[L^{a_{r1}} M^{a_{r2}} T^{a_{r3}} \right] \end{aligned} \tag{6.6}$$

In order to obtain a dimensionless power product of the Q_j one may write

$$\Pi \left[L^o M^o T^o \right] = \left[\left(L^{a_{11}} M^{a_{12}} T^{a_{13}} \right)^{x_1} \cdots \left(L^{a_{r1}} M^{a_{r2}} T^{a_{r3}} \right)^{x_r} \right] \tag{6.7}$$

The exponents $x_1 \cdots x_r$ of the dimensionless product are solutions of the set of homogeneous algebraic equations

$$\sum_{j=1}^{r} a_{jk} x_j = 0 \qquad (k = 1,2,3) \tag{6.8}$$

From the rank of the matrix $\| a_{jk} \|$ and the number of variables Q_j one obtains the number of dimensionless products in the complete set.

Bridgman [2] has shown that any fundamental system of solutions of equation 6.8 furnishes exponents of a complete set of dimensionless products of the Q_j's. There is arbitrariness in the determination of a fundamental system of solutions of equation 6.8. The result can however be made unique by specifying that the matrix of the solution shall have the following form.

$$\left\| \begin{array}{cccccccccc} 1 & 0 & 0 & \cdots & 0 & 0 & x_{1,r-m+1} & x_{1,r-m+2} & \cdots & x_{1,r} \\ 0 & 1 & 0 & \cdots & 0 & 0 & x_{2,r-m+1} & x_{2,r-m+2} & \cdots & x_{2,r} \\ 0 & 0 & 1 & \cdots & 0 & 0 & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \ddots & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \ddots & \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 1 & x_{r-m,r-m+1} & x_{r-m,r-m+2} & \cdots & x_{r-m,r} \end{array} \right\| \qquad (6.9)$$

Each row of the matrix, which incidentally represents a particular solution of equation 6.8, is a set of exponents in a dimensionless product of the Q's. The complete set of dimensionless products determined by this particular matrix construction has the property that each of the variables $Q_1 \cdots Q_r$ occurs in only one dimensionless grouping. This property has the advantage for the experimentalist in that it permits him to vary a specific dimensionless product while he can keep all others constant. This facilitates the study of the importance of a specific dimensionless grouping in a physical phenomenon as well as the representation of experimental data by graphical means.

6.4 APPLICATION TO UNDERGROUND EXPLOSIONS

The determination of the model laws for underground explosions requires a decision as to what variables enter into the problem. If variables are considered which do not really affect the phenomenon too many dimensionless groupings will appear in the final equations. If essential variables are omitted the final equations may not describe the phenomenon correctly. The problem as to what the necessary and appropriate variables are rests basically on the following factors.

Obviously one requires enough knowledge about the problem on either theoretical or experimental grounds to decide which variables influence the phenomenon. For example if the appropriate differential equations are known one can immediately determine the proper variables. Unfortunately a successful theoretical model is unavailable at this time. On the other hand the experimentalist has supplied us with a number of parameters which appear to be of importance in the description of the explosion

phenomenon. These are listed later.

The second factor to be considered is that the choice of variables and the test of their importance is in many instances governed by expediency (military, economic, etc). If say variables descriptive of the soil characteristics are chosen it is well to keep in mind that experimental checks as to their importance may well be impossible. Choice of test sites have in the past been based on other considerations than appropriate soil conditions [12].

In the choice of the appropriate parameters we have been guided by the experiments of C. W. Lampson [7] and E. B. Doll [4]. The phenomenon of underground explosion has been experimentally described in terms of peak values of the pressure $p[ML^{-1}T^{-2}]$, the particle acceleration a $[LT^{-2}]$, particle velocity v $[LT^{-1}]$ and particle displacement d [L]. We must restrict ourselves in the following to positive values of the peak parameters since by theorem 6.1, the Q_j's are assumed to be positive. The scaling of a complete wave profile is open to question. The parameters are determined as functions of the distance r [L] from the center of the explosion and the time t [T]. It is found that the phenomenon ddpends on the mass of the explosive W [M], a characteristic speed $\bar{c}$ $[LT^{-1}]$ with which a signal of small intensity travels through the mediums as well as on the density ρ $[ML^{-3}]$ of the ground It has furthermore been found that the decay of say the peak quantities depends ultimately on the depth s[L] of burial of the explosive below the surface. Shallow surface explosions are commonly coupled with air blast effects. The latter despite their short duration are of considerable importance since they introduce undesirable scale effects [4]. An additional parameter is needed to describe the effectiveness $E[M^oL^oT^o]$ of the blast producer. The explosive characteristics have been rated in the past with T.N.T. as a base. It is open to question at this time if the effectiveness E can be successfully correlated with the physico-chemical characteristics of the explosives.

In addition to the above we propose a parameter which we feel describes in many respects the energy degradation in <u>deep</u> underground explosions. Such a parameter appears to us of importance since it should characterize the maximum

stresses and strains which might arise in the medium. If for example a large portion of the explosive energy is utilized in creating a cavity or fissures in the soil, the residual stresses in the soil may have fallen to such small values that the medium conceivably exhibits elastic characteristics. (This of course can also be caused by a shallow burial explosion in which an appreciable amount of energy spends itself in air blast effects.) On the other hand if the medium resists the creation of a cavity the residual stresses may remain large enough for the medium to exhibit plastic characteristics. An appropriate parameter is perhaps the mean work per unit volume $Q\left[ML^{-1}T^{-2}\right]$ which is needed to expand the shot hole to a final state of rest. It has been found, in an analysis of the cavitation in which spheres were fired into clays, that the mean work per unit volume was constant over a wide range of striking velocities [6]. This seems to indicate that Q represents a characteristic soil parameter which is apparently independent of the intermediate time variation of the shot hole radius. Last it has been assumed by previous investigators that the environment of the experiment (i.e. the air and earth) is homogeneous. In the case of air this is a valid assumption but for the earth this may not necessarily be the case. If local soil variations exist proper account of these must be taken in modeling of experiments [4].

The relationships between the dimensionless grouping are readily derived from the above considerations. Consider as a specific example the following functional relationship.

$$f(p, r, s, t, Q, E, W, \bar{c}, \rho) = 0 \qquad (6.10)$$

The dimensional matrix is

$$\begin{array}{c|ccccccccc} & (1) & (2) & (3) & (4) & (5) & (6) & (7) & (8) & (9) \\ & p & r & s & t & Q & E & W & \bar{c} & \rho \\ \hline M & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ L & -1 & 1 & 1 & 0 & -1 & 0 & 0 & 1 & -3 \\ T & -2 & 0 & 0 & 1 & -2 & 0 & 0 & -1 & 0 \end{array} \qquad (6.11)$$

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The rank of the matrix is three and since there are nine variables, the number of dimensionless products is six. The exponents of the variables which form the dimensionless products are the solutions of the following set of equations:

$$\begin{aligned} x_1 + x_5 + x_7 + x_9 &= 0 \\ -x_1 + x_2 + x_3 - x_5 + x_8 - 3x_9 &= 0 \\ -2x_1 + x_4 - 2x_5 - x_8 &= 0 \end{aligned} \tag{6.12}$$

Solving these equations for x_7, x_8 and x_9 there results

$$\begin{aligned} x_7 &= -\tfrac{1}{3} x_2 - \tfrac{1}{3} x_3 - \tfrac{1}{3} x_4 \\ x_8 &= -2x_1 + x_4 - 2x_5 \\ x_9 &= -x_1 + \tfrac{1}{3} x_2 + \tfrac{1}{3} x_3 + \tfrac{1}{3} x_4 - x_5. \end{aligned} \tag{6.13}$$

With

$$x_1 = 1; \quad x_2 = x_3 = x_4 = x_5 = x_6 = 0$$

the equations 6.13 yield

$$x_7 = 0; \quad x_8 = -2; \quad x_9 = -1.$$

With

$$x_2 = 1; \quad x_1 = x_3 = x_4 = x_5 = x_6 = 0$$

these results

$$x_7 = -\tfrac{1}{3}; \quad x_8 = 0; \quad x_9 = \tfrac{1}{3}.$$

Continuing this process one can construct the matrix of the solution which has the following form

	(x_1) p	(x_2) r	(x_3) s	(x_4) t	(x_5) Q	(x_6) E	(x_7) W	(x_8) $\bar{c}$	(x_9) ρ	
π_1	1	0	0	0	0	0	0	-2	-1	
π_2	0	1	0	0	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	
π_3	0	0	1	0	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	(6.14)
π_4	0	0	0	1	0	0	$-\frac{1}{3}$	1	$\frac{1}{3}$	
π_5	0	0	0	0	1	0	0	-2	-1	
π_6	0	0	0	0	0	1	0	0	0	

The dimensionless products are then

$$\pi_1 = p(\bar{c})^{-2}\rho^{-1} \qquad \pi_4 = tW^{-\frac{1}{3}}\rho^{\frac{1}{3}}\bar{c}$$

$$\pi_2 = rW^{-\frac{1}{3}}\rho^{\frac{1}{3}} \qquad \pi_5 = Q(\bar{c})^{-2}\rho^{-1} \tag{6.15}$$

$$\pi_3 = sW^{-\frac{1}{3}}\rho^{\frac{1}{3}} \qquad \pi_6 = E$$

According to theorem 6.1, there results consequently a relationship of the form

$$\varphi\left(\frac{p}{\rho\bar{c}^2},\ r\left(\frac{\rho}{W}\right)^{\frac{1}{3}},\ s\left(\frac{\rho}{W}\right)^{\frac{1}{3}},\ t\bar{c}\left(\frac{\rho}{W}\right)^{\frac{1}{3}},\ \frac{Q}{\rho\bar{c}^2},\ E\right) = 0. \tag{6.16}$$

With

$$r\left(\frac{\rho}{W}\right)^{\frac{1}{3}} = \lambda_{r*}\ ;\quad s\left(\frac{\rho}{W}\right)^{\frac{1}{3}} = \lambda_{s*} \tag{6.17}$$

and assuming the possibility of solving equation 6.16 for the group $(p/\rho\bar{c}^2)$ we obtain for the peak pressure

$$p = \rho(\bar{c}\ )^2 f\left(\lambda_{r*},\ \lambda_{s*},\ t\bar{c}\left(\frac{\rho}{W}\right)^{\frac{1}{3}},\ \frac{Q}{\rho\bar{c}^2},\ E\right). \tag{6.18}$$

By similar arguments one obtains the general functional equations for the positive peak values of the particle

acceleration, velocity and displacement:

$$a = (\bar{c})^2\left(\frac{\rho}{W}\right)^{\frac{1}{3}} g\left(\lambda_{r*}, \lambda_{s*}, t\bar{c}\left(\frac{\rho}{W}\right)^{\frac{1}{3}}, \frac{Q}{\rho\bar{c}^2}, E\right) \tag{6.19}$$

$$v = \bar{c}\, h\left(\lambda_{r*}, \lambda_{s*}, t\bar{c}\left(\frac{\rho}{W}\right)^{\frac{1}{3}}, \frac{Q}{\rho\bar{c}^2}, E\right) \tag{6.20}$$

$$d = \left(\frac{W}{\rho}\right)^{\frac{1}{3}} \ell\left(\lambda_{r*}, \lambda_{s*}, t\bar{c}\left(\frac{\rho}{W}\right)^{\frac{1}{3}}, \frac{Q}{\rho\bar{c}^2}, E\right). \tag{6.21}$$

The appropriate model laws and scale factors are readily determined from equations 6.18 through 6.21.

The dimensionless parameters λ_R, λ_s have been used in the past to correlate experimental results. These parameters are related to λ_{r*} and λ_{s*} by setting $\rho = 1$ in the latter expressions. The correlation of experimental results for different soils is undoubtedly effected by this choice.

Special forms of the general functional equations 6.18 through 6.21 for the peak parameter, for example polynomial expressions in λ_r, have been used in correlating experimental data for both deep as well as shallow underground explosions with considerable success [7], [4].

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Project 1.9-1

APPLICATION OF THE KIRKWOOD-BRINKLEY METHOD TO THE THEORY OF UNDERGROUND EXPLOSIONS

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ILLUSTRATIONS

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ABSTRACT

The complete set of partial differential equations governing the flow of the medium behind the spherically symmetrical shock front has been reduced by Kirkwood and Brinkley to a pair of ordinary differential equations.

One of the most obvious difficulties in applying the Kirkwood-Brinkley method to the theory of underground explosions, as outlined here, is that the radiation effects in the initial, high-pressure phase of the explosion are not taken into account. On the other hand, the method has the advantage of providing a direct attack on the problem of underground explosions, since it presupposes only data which are experimentally measurable. However, it should be pointed out that the effort involved in the actual numerical integration of the two Kirkwood-Brinkley differential equations is trifling compared with the effort in constructing the tables for the Hugoniot function and Kirkwood-Brinkley enthalpy

Before applying the method to an earth medium, its application to air, as an intermediary step, would be desirable. In this way, the solutions could be compared with results that have been carried out by other methods, and also serve as a guide in developing the more complex underground case.

APPLICATION OF THE KIRKWOOD-BRINKLEY METHOD TO THE THEORY OF UNDERGROUND EXPLOSIONS

1.1 INTRODUCTION

The problem of predicting the sequence of events in an underground explosion depends ultimately on the solution of the partial differential equations of hydrodynamics, subject not only to proper initial conditions, but also to a moving boundary constraint represented by the Huguniot conditions at the shock front in the earth. A very serious problem arises from the difficulty in writing an equation of state for the earth medium. On the basis of a rough pressure-density curve for earth and an assumption on the energy, Griggs(1) has estimated shock velocities and peak pressures in an underground explosion for a plane shock. Unfortunately, it is difficult to estimate the effects of the approximations made, and the method does not admit of obvious extension toward greater accuracy.

1.2 THE KIRKWOOD-BRINKLEY DIFFERENTIAL EQUATIONS

The complete set of partial differential equations governing the flow of the medium behind the shock front (of planar, cylindrical, or spherical symmetry) has been reduced by Kirkwood and Brinkley(2) to a pair of ordinary differential equations. These differential equations (which are exact) are, in the case of spherical symmetry,

$$\frac{dp}{dR} = - \nu R^2 \frac{p^3}{D} M(p) - \frac{p}{R} N(p) , \qquad (1a)$$

$$\frac{dD}{dR} = - R^2 L(p) , \qquad (1b)$$

where R, the radius of the shock front, is the independent variable. The dependent variable p is the overpressure at the shock front (the pressure in excess of the pressure p_0 of the undisturbed medium). The dependent variable D is a quantity such that a non-vanishing value of its gradient implies an entropy increment of the medium due to passage of the shock; it will not be particularised further since it is defined by Eqs. (1) as a function of R. The functions L(p), M(p), N(p) are defined by

(1) See Project 1.9-2, JANGLE Report Series
(2) S. R. Brinkley, Jr. and J. G. Kirkwood, Phys. Rev. 71, 606 (1947).

$$L(p) = \rho_o\, h(p), \tag{2a}$$

$$M(p) = \frac{1}{\rho_o U^2}\, \frac{G}{2(1+g)-G}\,, \tag{2b}$$

$$N(P) = \frac{4(\rho_o/\rho) + 2(1-\rho_o/\rho)\, G}{2\,(1+g)-G}\,, \tag{2c}$$

in which

$$G = 1 - \left(\frac{\rho_o U}{\rho c}\right)^2, \qquad g = 1 - \frac{p}{u}\,\frac{dU}{dp}, \tag{3}$$

where U is the velocity of the shock front, ρ_o is the density of the undisturbed medium, ρ is the density of the disturbed medium at the shock front, c is the sound velocity $[(\partial p/\partial \rho)_s]^{\frac{1}{2}}$ of the disturbed medium at the shock front, and $h(p)$ is the Kirkwood-Brinkley enthalpy change (characterized in more detail later). The quantities $L(p)$, $M(p)$, $N(p)$ can be expressed as functions of the overpressure p in the disturbed medium at the shock front by virtue of the Hugoniot relations and the equation of state of the medium. The quantity ν is defined by

$$\nu = \int_0^{\infty} \frac{r^2(R,\tau)\; p'\,(R,\tau)\; u'\,(R,\tau)}{R^2\; p(R,o)\; u\,(R,o)}\, d\tau, \tag{4}$$

where the reduced time τ is

$$\tau = -\left[\frac{\partial}{\partial t}\ln\left\{r^2\,(R,t)\; p'(R,t)\; u'\,(R,t)\right\}\right]_{t\,=\,t_o\,(R)} \cdot \left[t - t_o\,(R)\right], \tag{5}$$

in which r is the Euler coordinate of the particles which are at the shock position R at the time $t = t_o(R)$, p' is the overpressure (above P_o) and u' is the particle velocity in the region behind the shock front. Note that the denominator of the integrand in Eq. (4) is simply the peak value (at the shock front) of the numerator.

The integrand of ν in Eq. (4) involves quantities (indicated by primes) evaluated in the region behind the shock front, and thus ν cannot be determined without a knowledge of flow conditions behind the shock. The Kirkwood-Brinkley approximation (as distinct from the Kirkwood-Brinkley equations (1)) consists in assuming that this integrand is a function only of τ. Explicitly, if the integrand is taken

as $e^{-\tau}$, one has $\nu = 1$. An empirical value of slightly wider applicability is $\nu = 1 - \exp(p'/p_s)$. This approximation constitutes a similarity assumption, which makes Eqs. (1) depend only on quantities evaluated at the shock front.

Equations (1) contain p and D as dependent variables, with the radius R of the shock front as independent variable. To integrate these equations for a general medium, one must express the two ancillary variables v ($\equiv 1/\rho$ = specific volume) and h as functions, v (p) and h(p) respectively, of p. Thus one must have:

(a) The equation of state for the medium.
(b) The Hugoniot function for the medium, which fixes the specific volume v(p) at the shock front.
(c) The Kirkwood-Brinkley enthalpy function h(p), which appears explicitly in Eq. (2a).

The Hugoniot condition is obtained from the conservation conditions at the shock front and is

$$H(p,v) = e(p,v) - e(p_0,v) + (v-v_0)\,\frac{p + p_0}{2} = 0 \qquad (6)$$

where e is the specific internal energy of the disturbed medium at the shock front and zero subscrips refer to the undisturbed medium. To fix v = v(p) from the Hugoniot condition, it is necessary to know the specific internal energy e as a function of pressure and specific volume. Since e is a function of two independent state-coordinates, in general, its tabulation requires a two-argument table (that is, a book). The relation e = e(p,v) amounts to an equation of state. Note that the Hugoniot condition is likewise essential to specify the quantity $[(\partial p/\partial \rho)_s]^{1/2}$ (which appears explicitly in G of Eq. (3)) as a function of p. To specify the Kirkwood-Brinkley enthalpy function h(p), consider Figure 1. The (irreversible) shock front moves the initial state-point (p_0, v_0) of the medium up the Hugoniot curve to the point (p,v). After the shock has passed, the medium undergoes an adiabatic expansion from the point (p,v), ultimately reaching the initial pressure p_0 at the point (p_0, v_f). The Kirkwood-Brinkley enthalpy function h(p) is defined as the enthalpy increment in going from the point (p_0, v_0) to the point $(p_0, v_f(p))$. This function can be written as the integral

$$h(p) = \int_{T_0}^{T_f(p)} c_p(p_0, T)\, dT \qquad (7)$$

where c_p is the specific heat at constant pressure and T is absolute temperature. Alternatively, one can write from Figure 2,

$$h(p) = \int_{S_o}^{S_f(p)} T\,(p_o, S)\, dS \qquad (8)$$

where S is the entropy and S_f is its final value after the adiabatic expansion. The most convenient method of computing h(p) remains to be determined, and it seems worth while to explore other possibilities besides those given.

The thermodynamic data to determine the Hugoniot function and the Kirkwood-Brinkley enthalpy function for an earth medium can be obtained from:

(a) Measurements of Bridgeman and others(3) on compressibilities and expansion coefficients for various minerals, extending up to a pressure range of about 10^5 bars and covering a restricted range of temperature.

(b) The theory of a Fermi gas(4) applied to material under high compression (over about 10^7 bars).

The region 10^5 - 10^7 bars, not covered by the sources of data above, can at present be determined only by reasonable interpolation. However, recent discussions with Professor P. M. Morse have indicated some promise of extending the Fermi-Thomas theory down to this range of pressure.

The various methods of handling the initial conditions on the solution include (in order of probable usefulness):

(a) Taylor's similarity solution(5).
(b) Isobaric sphere behind shock front at an initial time.
(c) Point source solution (i.e. not of similarity type).

1.3 DIFFICULTIES IN APPLYING THE METHOD

It remains to point out some obvious difficulties in applying the Kirkwood-Brinkley method to the theory of an underground explosion. As outlined here, the method takes no account of radiation effects(6) in the initial, high-pressure phase of the explosion. At intermediate pressures, the possibility of plastic deformation and phase transitions in the earth medium causes complications whose effect is difficult to assess. At low pressures, the method presupposes a simple and adequate equation of state for an earth medium.

Before applying the method to an earth medium, it would be desirable, probably, to apply it to air. In the case of air, the necessary tables(7) over much of the requisite range are already available. Although shock calculations for air have been carried out by other methods, application of the Kirkwood-Brinkley method to air is of considerable interest, not only as a check but also as a guide in the more complex underground case.

(3) through (7) next page.

(3) F. Birch (Editor), Handbook of Physical Constants (Geological Society of America, Special Paper No. 36).
(4) J. C. Slater and H. M. Krutter, Phys. Rev. 47, 559 (1935); R. P. Feynman, N. Metropolis, and E. Teller, Phys. Rev. 75, 1561 (1949; W. M. Elsasser, Science, 113, 105 (1951); N. Metropolis and J. R. Reitz, J. Chem. Phys. 19, 555 (1951).
(5) G. I. Taylor, Prac. Roy. Soc. A 201, 159 (1950).
(6) Stanford Research Institute, Technical Report No. 1, Contract N7onr32104 (and Memorandum: Estimate of Revisions of Technical Report No. 1) Dec. 15, 1950 (SECRET-AEC RESTRICTED DATA).
(7) J. O. Hirschfelder and J. L. Magee, Report MDDC-590, U. S. Atomic Energy Commission (Declassified, January 1, 1947).

1.4 ADVANTAGE OF KIRKWOOD-BRINKLEY METHOD

The Kirkwood-Brinkley method has the advantage of providing a direct attack on the problem of underground explosions. It pre-supposes only data which are experimentally measurable, and its procedure is independent of analytical artifice for its execution. However, a salient feature of the method should be emphasized. The labor involved in the actual numerical integration of Eqs. (1) is trifling compared with that of constructing the tables for the Hugoniot function and Kirkwood-Brinkley enthalpy. As pointed out, one requires a book of tables (a two-argument table) to determine the Hugoniot function, and at least a single table to determine h(p). Furthermore, no solutions of the hydrodynamic equations are available until the tables are completed. This feature of the method is a handicap in applying it to a medium (such as earth) for which reliable asymptotic or approximate solutions are not (as yet) available. It should be noted, of course, that the need for these thermodynamic tables is not peculiar to the Kirkwood-Brinkley integration scheme, since all integration methods require essentially the same thermodynamic data. In similarity solutions, for example, the role of the tables is taken by a suitable approximation for γ (specific heat ratio).

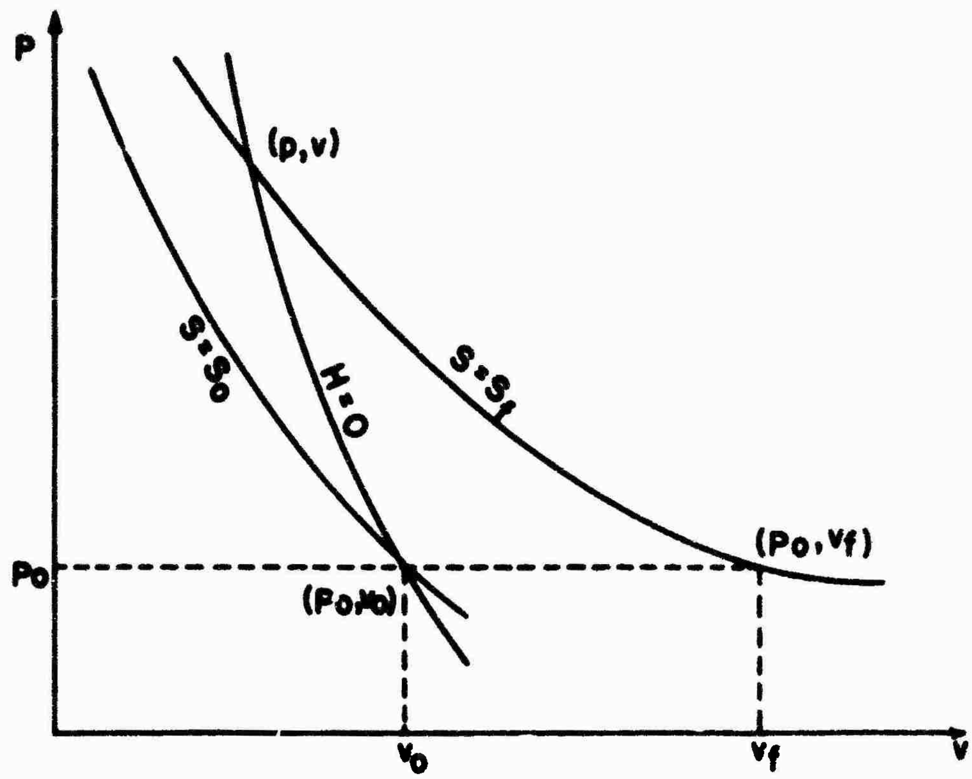

Figure 1 Specifying the Enthalpy Function (See Eq. 7)

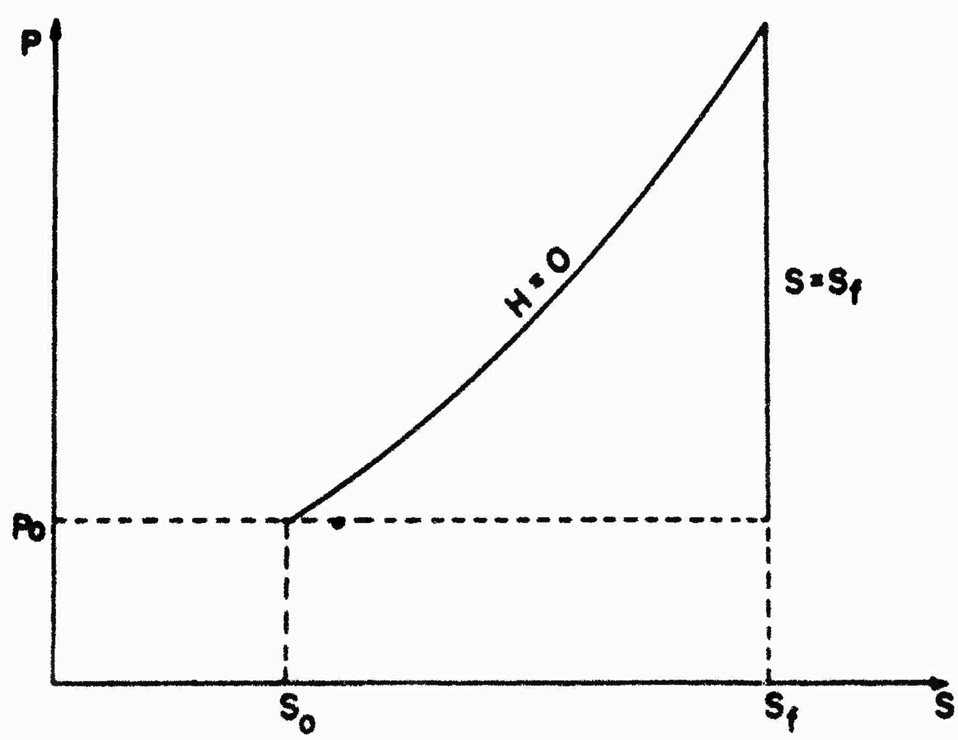

Figure 2 Alternately Specifying the Enthalpy Function (See Eq. 8)

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OPERATION JANGLE

Project 1.9-2

NOTES ON SURFACE AND UNDERGROUND BURSTS

By

D. T. GRIGGS

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RESTRICTED DATA
ATOMIC ENERGY ACT 1946

ACKNOWLEDGEMENT

The concepts employed in this report originated with Dr. Edward Teller. While he should not be held responsible for the whole treatment, he has aided materially in its execution.

PROJECT 1.9-2

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ILLUSTRATIONS

TABLES

PROJECT 1.9-2

ABSTRACT

A crude first order theory is applied to derive the shock conditions resulting from a point energy source in an infinite homogeneous medium having characteristics similar to soils. Approximate pressure-distance-time values are determined. Some comparisons are made between the effects of surface and shallow underground bursts.

NOTES ON SURFACE AND UNDERGROUND BURSTS

1.1 INTRODUCTION

The chief aim is to derive approximate values for the ground effects resulting from surface vs. shallow bursts. It seems clear that the key to this question lies in the early history of the explosion when the propagation in the earth is in the form of an intense shock wave.

It is concluded that a surface burst will produce about the same earth disturbance as a shallow burst (scaled to 50 feet burial for 25 KT) of approximately one-tenth the energy release.

1.2 APPROXIMATE SHOCK CONDITIONS IN A DEEP BURST

For a short time after a deep underground nuclear explosion the energy will be propagated outward as a true shock wave, since the energy of compression far exceeds the energy of distortion. The characteristics of this shock wave propagation can be derived approximately from the Hugoniot conditions, the equation of state of Feynman, Metropolis, and Teller for highest pressures, and compressibility measurements in the intermediate pressure range.

The Hugoniot conditions for conservation of mass and momentum across the shock front may be written in the following forms for a strong shock:

Page 1 - Equation (1) should read:

$$v_o = \sqrt{\frac{P}{\rho_o}\,\frac{\rho_1}{(\rho_1-\rho_o)}}$$

Corrected 5 Mar. 1953 G.E.B

Equation (2) should read:

$$\Delta E = \frac{P}{2}\left(\frac{\rho_1-\rho_o}{\rho_1\rho_o}\right)$$

The four lines following Equation (2) should read:

where: v_o = shock velocity, P = peak pressure in the shock, ρ_o = undisturbed density of the earth, ρ_1 = density immediately behind the shock front, ΔE = internal energy increment per unit mass across the shock.

where: v_o = s
disturbed dens
shock front, E
shock.

Feynman, Metropolis, and Teller's calculations for the appropriate atomic weight, Bridgman's data on compressibility, and rough estimates on the compaction of typical soil in the low pressure range yield a

relationship between density and pressure which is plotted in Fig. 1. From this and equation (1) the shock velocity may be derived as a function of pressure. This is plotted in Fig. 2. In the pressure range of about 100 to 1000 bars, the shock will be transformed into a distortional "earth pressure wave" of the type observed by Lampson. This transition is indicated very roughly by the dashed line on Fig. 2. The undisturbed soil density is assumed to be 1.8. For other initial densities, the velocity of an intense shock will vary as

ρ_o^{-3}. The shock velocity at a given pressure is not sensitive to the form of the ρ vs. P relation, so that the rough curve of Fig. 1

The shock velocity as a function of radius for a deep burst may be obtained to a rough approximation from the above and equation (2) by assuming that the energy per unit mass is constant within the shock front and that the total energy is:

$$E = 2VE\rho_o$$

where V is the volume inside the shock front. Shock velocity is plotted vs. radius for a 1.25 KT shot in Fig. 3. Integrating graphically, shock radius as a function of time is obtained and plotted in Fig. 4 (lower curve). For comparison, the shock radius of a similar shot in air is plotted (upper curve) scaled from data on the SANDSTONE X-ray shot.

1.3 COMPARISON OF SURFACE AND SHALLOW BURSTS

The propagation of a shock from a point source explosion at the earth-air interface may be derived to a rough approximation by assuming that the pressure is constant within the shock front. The initial condition for shock propagation may be taken as the end of the phase of predominant propagation by radiative transport. At this time the radius of the shock front in air is approximately 5 meters, while the penetration into the ground will be negligible due to the high opacity of the soil and the comparatively slow velocity of shock propagation. From this time until the pressure in the soil drops sufficiently so that propagation of energy occurs by the distortional plastic wave, the shock velocity in air remains much higher than that in ground. The pressure versus radius in air may then be approximated by Taylor's expression:

$$\frac{P}{P_o} = 0.155\left(\frac{E}{r^3}\right)$$

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PROJECT 1.9-2

The velocities in air, in ground, and their ratio are given in Table 1 as a function of radius for a 1.25 KT burst.

TABLE 1

Velocities As A Function of Radius

r (meters)	P (atm.)	v_a (km/sec)	v_g (km/sec)	$\frac{v_a}{v_g}$
5	1.3×10^5	110	4.0	27
10	1.6×10^4	39	1.5	26
15	$.48 \times 10^4$	21	0.84	25
20	$.20 \times 10^4$	14	0.56	25
30	6.0×10^2	7.5	0.3	25

v_a is calculated from the Hugoniot equations for an ideal gas with $T = 1.4$; v_g is obtained from Fig. 2. 3.

The ratio of the rate of doing work on the air and on the ground is:

corrected 5 Mar. 1953 - geb.

$$\frac{\dot{W}_a}{\dot{W}_g} = \frac{2\pi r^2 P v_a}{\pi r^2 P v_g} = \frac{2 v_a}{v_g} .$$

This ratio is nearly constant at 50:1 out to the point at which the propagation in earth is distortional in character. From this point on, the effect on the earth pressure wave of the shock wave in the air will be small since it will act only to prolong slightly the duration of the strain impulse. This reasoning then implies that the energy in the earth pressure wave from a surface burst will be about 2 per cent of that from a deep source of the same energy release. This is to be compared with about 20 per cent for a surface vs. deep TNT explosion.

A 1.25 KT explosion at a depth greater than 3 meters will produce

effects similar to an equally energetic TNT explosion at the same depth. The only important difference would appear to be the heat energy left in the initially vaporized earth when it has expanded to atmospheric pressure. Since the surface of the earth will be breeched while the shock wave is still intense, the energy coupled to the ground will depend on the loss of energy to the air rather than on the initial character of the source. It is estimated that some 20 per cent of the nuclear energy will be wasted in a shallow burst, as compared to an equally energetic TNT explosion.

Using Lampson's coupling factor as a measure of the energy coupled to the ground, and the above correction, a 1.25 KT shot buried 18 ft. would produce about 25 per cent as energetic an earth pressure wave as a deep 1.25 KT shot. The surface shot is thus estimated to be roughly one-tenth as efficient in coupling energy to the ground as a shot at a scale depth of 18 ft. for 1.25 KT or 50 ft. for 25 KT.

1.4 MISCELLANEOUS PHENOMENA

The internal energy increment ΔE at the shock front may be used as a criterion for estimating the radius of earth vaporization and melting for an underground shot. The heat to vaporize soil is approximately 3000 cal./gm., and to melt it, 400 cal./gm. $\Delta E = 3000$ $E = 3000$ cal./cal./ at a radius of 3 meters, and 400 cal./gm. at 6 meters. It thus appears that in the shallow shot, buried 5.6 meters, an incandescent fireball will just breech the surface. *Corrected 5 Mar 53 - 9eb*

It seems certain that both underground shots will produce sufficiently dense clouds to settle after a small initial rise, producing a "base surge." The height of rise will be somewhat less, and the rate of settling somewhat larger than for an equivalent TNT shot, due to the greater density of the final products in the nuclear case.

In the surface shot on the other hand, since 98 per cent of the energy is spent on the air, one would expect the history of the cloud to be not greatly different from a tower shot. The material ejected from the crater, however, will largely settle to the ground, and it may be expected to carry some 5-10 per cent of the fission products. The history of these fragments is not obvious to me because of the complications of the afterwind associated with the rising hot air mass.

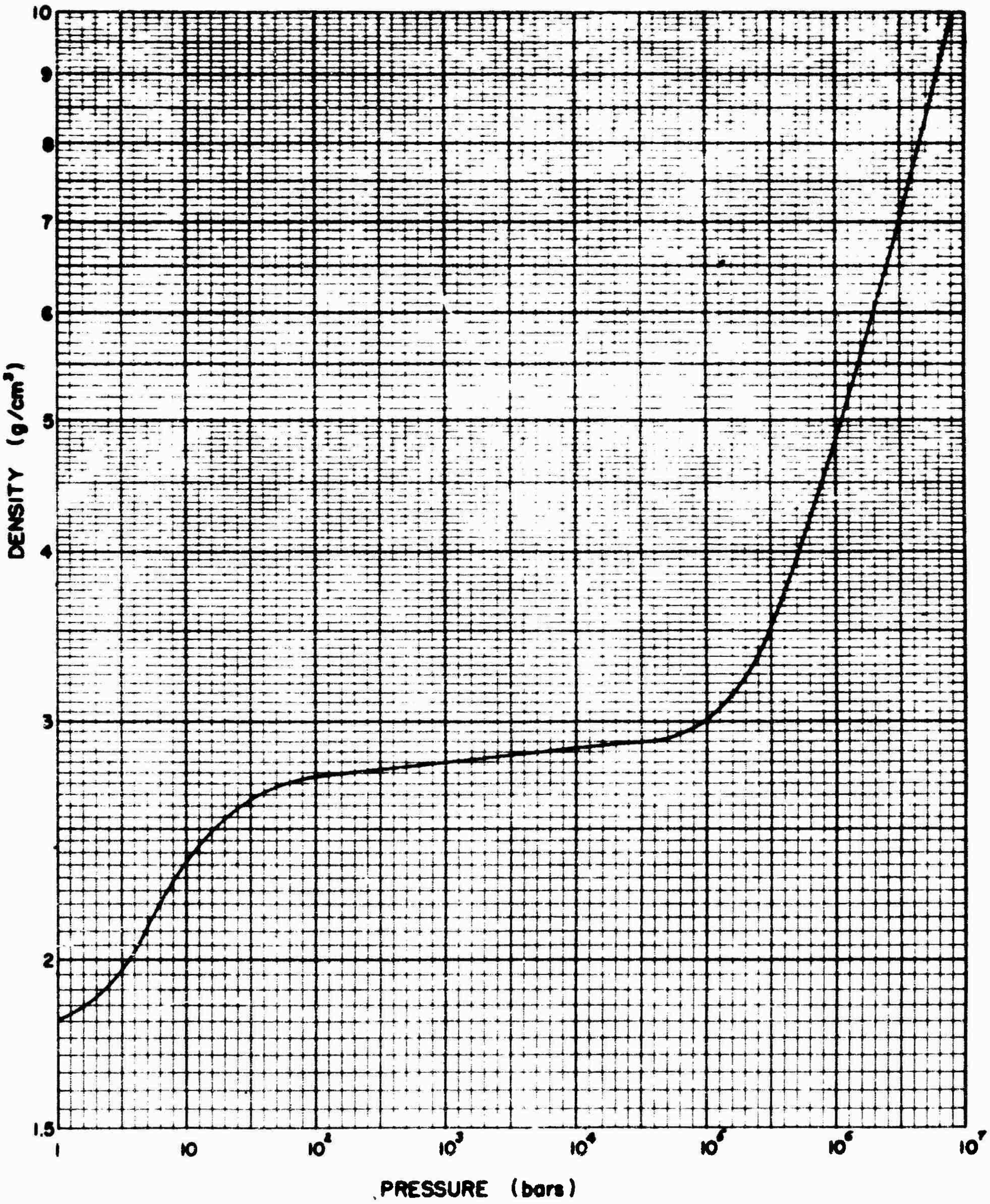

Figure 1 Relation Between Density and Pressure in a Typical Soil

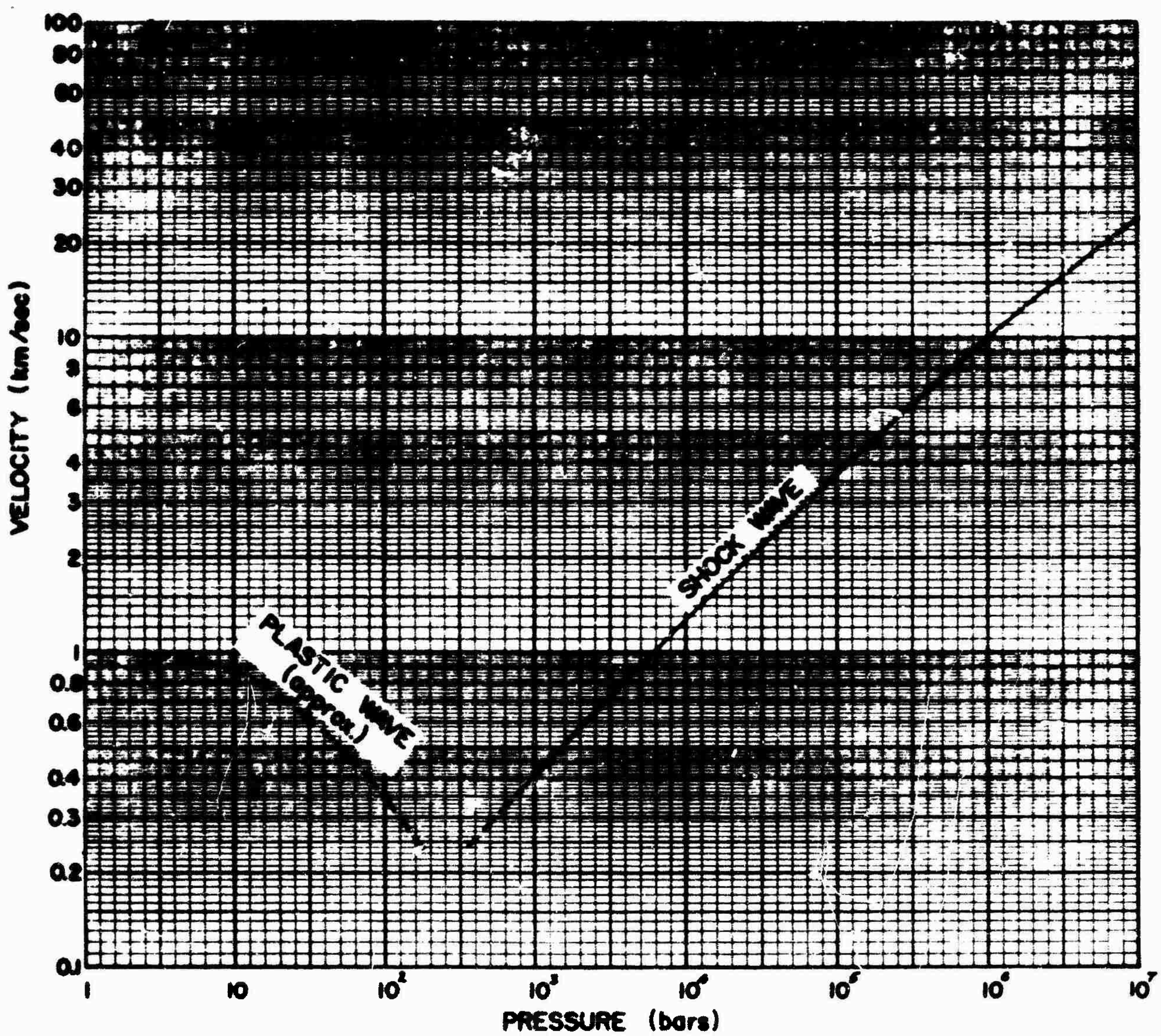

Figure 2 Shock Velocity as a Function of Pressure

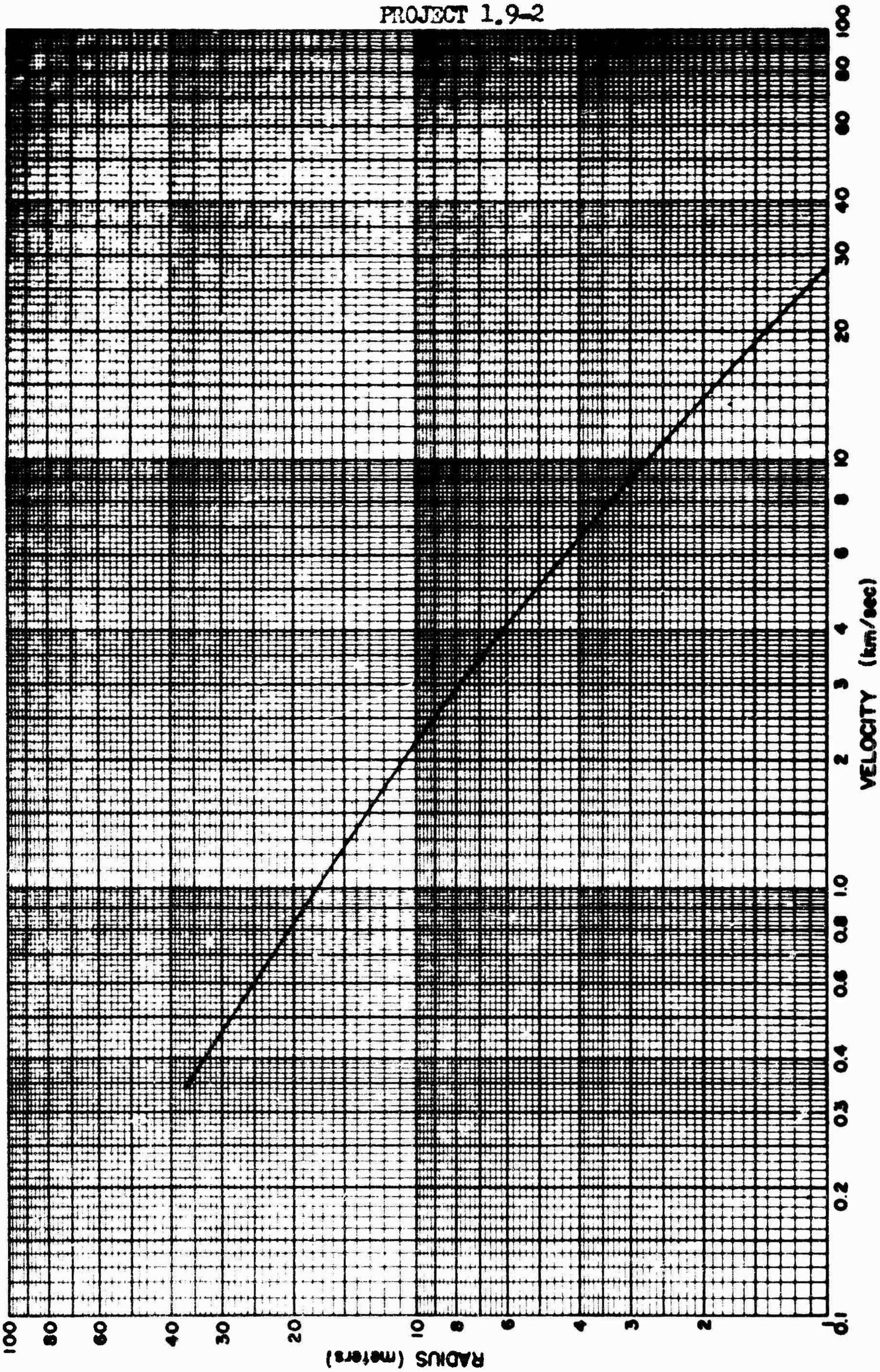

Figure 3 Shock Velocity vs Radius for a Deep Underground 1.25 KT Shot

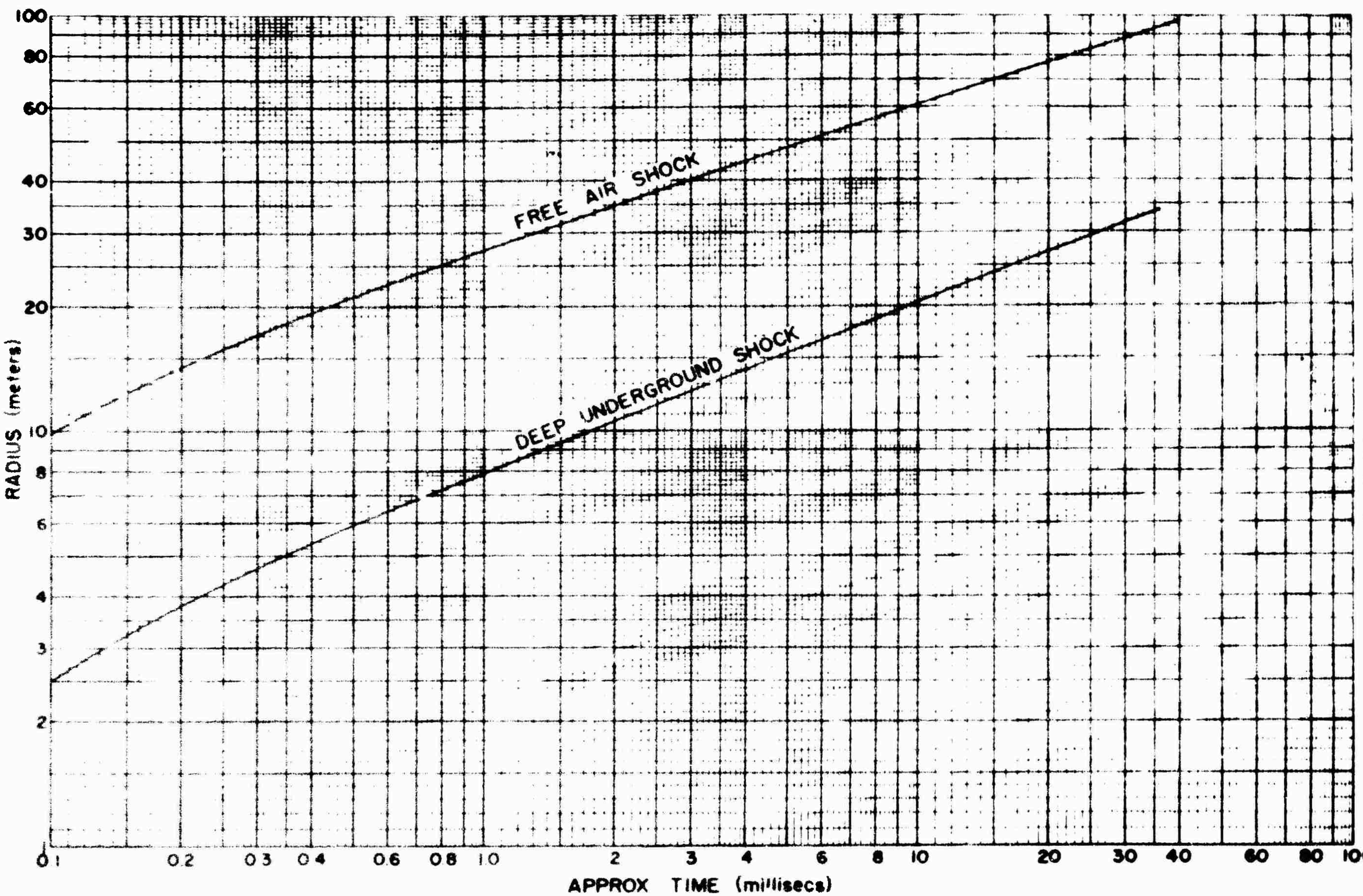

Figure 4 Comparison of Shock Radius for Air and Underground Bursts.

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OPERATION JANGLE

PROJECT 1.9-3

PREDICTIONS FOR UNDERGROUND TEST

By
Vincent Salmon

27 November 1951

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RESTRICTED DATA
ATOMIC ENERGY ACT 1946

PROJECT 1.9-3

PREFACE

The text of this report is substantially that of Technical Report No. 4, "Predictions - U Test - Operation JANGLE," by V. Salmon, November 8, 1951, prepared under Office of Naval Research Contract N7onr-32104. The substance of the present report differs from the former one by the addition of material used during an oral presentation at a pre-test symposium at Mercury, Nevada, on 27 November 1951.

The results are presented with a minimum amount of the analyses by which they were developed. A much more detailed, general, and complete report on phenomena associated with underground nuclear explosions is in preparation and will include a comparison with such results as are available.

PROJECT 1.9-3

CONTENTS

PROJECT 1.9-3

ABSTRACT

Results of an analytical treatment, together with condensed statements of assumptions, are presented with a minimum of analytical details for the mechanical phenomena ensuing from the underground detonation of a nuclear weapon. Conditions have been idealized by assuming the instantaneous release of 1 KT of energy (TNT equivalent) in a very small volume and in a dry silica soil. A modified form of Bethe's small $(\gamma - 1)$ theory is used to obtain rough numerical estimates of pressures, temperatures, velocities, and dimensions associated with the breakaway bubble, wave phenomena in the earth, venting of earth gas, air-blast energy, height of atomic cloud, throwout velocity, breakup of surface targets, return of throwout, and energy partition.

CHAPTER 1

INTRODUCTION

1.1 SCOPE OF STUDY

A general study has been made of underground nuclear explosions by tracing the phenomena from their very early stages out to the final effects. This report presents the predictions resulting when the analysis is applied to the Underground Test of Operation JANGLE. In addition, the values obtained are compared, whenever possible, with results obtained from other analyses or from extrapolation of data from HE explosions. Confidence limits for the results are not known; a two-to-one uncertainty is not out of the question. The analysis should be considered an introductory guide to the phenomena, which may be studied more carefully when sufficient experimental data become available. This study embraces mechanical phenomena only; thermal and nuclear radiation are not considered.

The explosion phenomena are considered under six headings:

1. Breakaway of pressure wave from gas bubble.

2. Transmission (wave) phenomena.

3. Venting of gases.

4. Throwout and missiles.

5. Return to earth of material thrown out.

6. Energy partition.

1.2 TEST CONDITIONS ASSUMED

The nuclear explosion is assumed to have a total energy release of 4.2×10^{19} ergs, equivalent to that from one kiloton of TNT. Of this energy, 15 per cent is assumed to be in delayed radioactivity not available for prompt mechanical effects. The gadget is idealized to a source of energy sufficiently small so that point source theory may be used. The effective center of the explosion is taken at 17 feet below the surface of a sandy soil, which is assumed homogeneous for the most part. The soil is assumed to have a porosity of 30 per cent and to be 25 per cent saturated. The solid constituent is assumed to

be wholly silica of voidless density 166 lb/ft^3. At depths below 10 feet the unit weight of soil is taken as 110 lb/ft^3, with a seismic velocity of 3000 fps. For purposes of predicting certain anomalies expected in the free-earth phenomena, a seismic velocity of 4500 fps is assumed to exist below a depth of 100 feet.

For cloud rise calculations, the average cloud altitude above sea level is taken as 10,000 feet, and the characteristics of the US Standard Atmosphere are evaluated at that altitude.

CHAPTER 2

PREDICTIONS

2.1 BREAKAWAY

When the energy released by the nuclear detonation reaches the earth surrounding the weapon, the energy density is so great that the earth is quickly converted to a highly energetic mixture of nuclear particles, ions, atoms, and photons. At this stage particle and radiation actions and interactions are inelastic in the sense that the boundary containing the energy grows by converting the solid material exterior to it to more particles of the same type, rather than by outward radial motion of single particles. The material within the boundary will be called earth gas. Eventually, however, the energy density falls so much that elastic impacts appear, become more numerous, and finally predominate. Somewhere in this process the surrounding earth is finally able to transmit elastic wave signals faster than the boundary grows by engulfing material. At this stage the pressure wave breaks away from and outruns the earth-gas bubble, which from then on grows principally by the outward motion of the particles in its boundary.

As used here, breakaway is arbitrarily defined by the conditions for which 50 per cent of the oxygen atoms are singly ionized. The precise conditions may be evaluated by a modified application of Bethe's small $(\gamma - 1)$ theory. Here γ is the exponent in the equation of state of earth gas,

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} . \qquad (1)$$

The results of this analysis, when applied to the test in question, depend on the energy release appearing as radioactivity. Since this represents energy not promptly available for mechanical effects, it is subtracted from the total release to give the energy with which we are concerned. In the absence of a complete evaluation of GREENHOUSE data, the prompt (mechanical) energy has been taken as the earlier figure of 85 per cent of the total release.

The analysis then predicts the following parameters of the gas bubble at breakaway:

$\gamma = 1.64$

Temperature $T_b = 220{,}000^\circ$ K

Pressure $p_b = 3.2 \times 10^8$ psi

Relative density at boundary $\rho/\rho_o = 4.1$

Shock velocity $U_b = 1.3 \times 10^5$ fps

Material velocity $u_b = 1.0 \times 10^5$ fps

Time to reach breakaway $t_b = 12 \mu s$

Weight of earth $W_b = 5.6$ tons

Radius of bubble $r_b = 2.9$ ft

There is reason to believe that this calculated pressure is high and the breakaway radius low. The analysis implicitly assumes that the size and weight of earth affected is considerably greater than that of the gadget. It is probable that this assumption will not be valid at this epoch.

The value of t_b is obtained from the Neumann-Fuchs-Taylor relation for point explosions.

It is of interest that at breakaway, radiation comprises about 0.001 per cent of the energy released. Its importance as an energy transport mechanism at breakaway is greater than this value indicates, and may be estimated by the following. Assume the temperature at breakaway to remain constant for the ensuing 0.1 millisecond. Using 2700 calories/gm as the energy to vaporize (under ordinary conditions) earth from the solid, calculate the engulfment increase in radius from the radiation flux; the value turns out to be one foot. However, in the same time, the material velocity would have advanced the radius 10 feet. Hence it is seen that radiation transport in the earth is not too important after breakaway.

2.2 TRANSMISSION PHENOMENA

We are here concerned with the pressure in the true earth shock leaving the bubble of earth gas, and its subsequent decay. A subsidiary result is a theory of crater formation founded on Westergaard's ideas.

It is first necessary to form a rough equation of state for highly compressed earth. In the Tait form this is

$$\frac{p + p_i}{p_i} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} . \tag{2}$$

The exponent γ and the internal pressure p_i have been estimated in the following manner. Compare the isentropic and isothermal equations of state for water. From Bridgman's data on the compressibility of substances similar to highly compressed earth constituents (silica, glass, rock), form an isothermal equation of state. Assume that the isentropic constants for the solids are related to the isothermal ones in the same ratios as for water. This gives as estimates

$$\begin{aligned} p_i &= 19{,}000 \text{ psi} \\ \gamma &= 84 \end{aligned} \tag{3}$$

The high value of γ corresponds to the high incompressibility of earth, once the voids have been removed. These constants give a seismic velocity of some 6700 fps; for sandstone the velocity lies between 4600 and 14,000 fps. This indicates that the values for the constants p_i and γ are not too unreasonable.

From this equation of state, and assuming continuity of pressure and material velocity, the pressure in the shock delivered to the earth at breakaway can be calculated. It turns out to be 6.4×10^8 psi. It is also of interest that since the earth is stiffer than the earth gas, at breakaway a positive pressure pulse goes radially backwards into the gas bubble. Almost complete reflection obtains, indicating the "mismatch" between source (gas bubble) and load (the earth). In fact, the shock-acoustic impedance $\rho_0 U$ in earth is about nine times that in the gas bubble at breakaway.

The initial earth shock is assumed to decay according to isothermal sphere explosion theory until the "TNT radius" of 17 feet has been reached. Then a $1/r^3$ law is joined on, yielding an expression for the peak pressure

$$p = \frac{16 \times 10^9}{r^3} \text{ (psi, ft)} \tag{4}$$

Note that the early infinite-earth shock pressure has been identified with the later peak of a wave form degenerated by reflection, hysteresis, and plastic flow effects. If the earth pressure results from

the HE-2 test[1*] are corrected for gage depths and are extrapolated to a one kiloton explosion, there results

$$p = \frac{2 \times 10^9}{r^3} \quad \text{(psi, ft)} \tag{5}$$

This is about one-eighth the pressure predicted from nuclear considerations, and may indicate the effect of the reflections which were neglected in that analysis. The pressures are expected to lie closer to the values extrapolated from the HE-2 test than to those predicted from the breakaway analysis. This tends to confirm the suspicion that the above breakaway parameters, computed from Bethe's small $(\gamma - 1)$ theory, are somewhat too energetic.

Expressions have been obtained for the compressional, kinetic, and plastic-flow energy per unit volume in the pressure shock wave. These are respectively

$$e_c = p_1 \left\{ \frac{1}{\gamma - 1} \left[\left(1 + \frac{p}{p_1}\right)^{1 - \frac{1}{\gamma}} - 1 \right] + \left[\left(1 + \frac{p}{p_1}\right)^{-\frac{1}{\gamma}} - 1 \right] \right\}, \tag{6}$$

$$e_k = \frac{p}{2} \left\{ \left(1 + \frac{p}{p_1}\right)^{\frac{1}{\gamma}} - 1 \right\}, \tag{7}$$

$$e_p = \frac{2Y}{3} \left\{ 1 + \ln \frac{E}{3Y(1 - \nu)} \right\}. \tag{8}$$

In Equation 8, Y is the dynamic compressive yield stress, E is Young's modulus, and ν is Poisson's ratio for earth.

When the pressure-distance relation (5) and the parameters of (3) are substituted, it turns out that $p = p_1$ at $r = 46$ feet, and that at this distance the compressional energy density is only seven per cent below the kinetic. At lower pressures they approximate $p^2/2\gamma p_1$, until at about 130 feet both are equal to the plastic flow energy density of about 5×10^8 erg/ft^3 for JANGLE soil. These calculations are

* Superscript numbers refer to references given in the Bibliography at the end of the report.

uncertain to the extent that the pressure law and the effect of high pressure on earth "viscosity" are unknown. The latter in particular is questionable, since it is known, for example, that at extremely high pressures, lubricating oil has practically the viscosity of copper at ordinary conditions. Hence plastic flow will probably use more energy than anticipated, especially near the charge.

At $p = p_1$, at $r = 46$ feet, the predicted earth shock velocity is 8200 fps. Thus the seismic regime exists throughout the region in which measurements are possible.

An approximate theory of cratering has been developed, based on Westergaard's ideas of the tension wave resulting from the reflection of a pressure wave at the surface of the earth. It is assumed that the boundary of the real crater is the locus of points for which the peak magnitude of the tension wave is equal to the sum of the dynamic tensile strength of earth plus the geostatic pressure. The relation is

$$\{r^2 + (h + z)^2\}^{\frac{n}{2}} \{p_t + \rho gz\} = A, \qquad (9)$$

where r and z are the boundary coordinates, h is the depth of charge, p_t is the tensile strength of earth, ρ is the density of earth, and the constants n and A arise from the pressure relation $p = A/r^n$. Owing to the disturbance created by the wave, it is probably a good approximation to assume that the earth within the crater has been so decohered that it acts as a viscous liquid. This partially justifies the use of the geostatic pressure without modification by the elasto-plastic properties of soil. In order to obtain the tensile strength of JANGLE earth, an estimate was made of the radius of the real crater for the HE-2 shot; extrapolated to a one kiloton shot, this is 180 feet. The value then obtained for the tensile strength was 340 psi. It is of interest that the dynamic tensile strength of concrete is about 300 psi, indicating that either the analysis is incorrect or else the dynamic tensile strength is much greater than the static (this is true for water).

When this tensile strength is used in the theory to predict the crater depth, a value of 150 feet is obtained. This is considerably in excess of the 62.5 feet estimated by extrapolation from HE-2. A partial explanation for the large discrepancy is that the crater radius predicted by the theory is not that of the real crater, owing to collapse of the walls near the surface. However, if the depth is to be predicted as equal to that from HE extrapolation, then the effective radius to be used will be less than half that extrapolated,

and the tensile strength obtained will be unreasonably high. In view of this, the extrapolated predictions are probably more reliable. To repeat, the estimated true crater diameter is 360 feet and the depth 62 feet.

It is known that the earth at the site shows a fairly rapid change of seismic velocity at depths around z_a = 100 feet. By standard geophysical calculations (knowing the velocities v_1 and v_2 in the upper and lower layers respectively), it is predicted that beyond a distance r_a, transmission will be primarily from a path down to the interface, along the interface, and up to the surface. The distance is given by

$$r_a = 2z_a \left(\frac{v_2 + v_1}{v_2 - v_1} \right)^{\frac{1}{2}} . \qquad (10)$$

Beyond r_a the direction of arrival should be predominantly vertical. Also, since seismic energy trapped by the wave-guide action of the interface spreads in two dimensions only, this path should have less attenuation than the direct one. Thus it is predicted that beyond r_a the vertical component of the acceleration should decay much less rapidly than nearer the charge. With v_1 = 3000 fps and v_2 = 4500 fps, we get r_a = 450 feet, or $\lambda \cong 3.6$. In the HE-1 and HE-2 shots, this phenomenon was actually observed. The "turn-over" distances were about 480 feet and 440 feet respectively. The distance for the underground nuclear test will of course depend on the actual underground profile, but 450 feet appears to be a good working value.

Values for the acceleration have not been predicted in this report, for no theory of spherical transmission in a finite elasto-plastic earth has been developed. Ordinarily it is assumed that the acceleration is proportional to the pressure gradient, which would make the exponent in the acceleration law equal four for a $1/r^3$ pressure law. Actually, in the HE shots, most of the exponents were near two. No explanation is advanced to explain this, although the viscosity term in the Navier-Stokes relations may be large enough to account for the difference. Values extrapolated from HE experience should be used.

When the pressure wave hits a horizontal target lying on the surface, such as a reinforced concrete highway, it is of interest to estimate the effect of shattering. An analysis based on Newmark's theory of crack width in shock-loaded concrete beams has been devised. If the energy delivered to the concrete is supposed to be equal to the kinetic energy density in the earth shock, there results

$$r_s = \left(\frac{2\alpha t^2 w A^2}{\sigma A_s d \gamma p_1 \epsilon} \right)^{\frac{1}{2n}} . \qquad (11)$$

Here r_s is the limiting distance inside which all reinforcing bars are ruptured; α is the ratio (distance from neutral zone)/(thickness); t is the thickness; w is the width (here 10 feet); A and n are the constants in the pressure relation $p = A/r^n$; σ is the plastic strength of the reinforcing steel; A_s is the area of the steel; d is the depth of the tensile steel; γ and p_1 are the constants in the equation of state of earth; and ϵ is the strain on the slab surface which will rupture the steel.

When the constants appropriate to the underground nuclear test are inserted, it turns out that r_s is about 70 feet. Thus inside this radius the concrete will probably be in pieces less than six-inch cubes, the spacing of the reinforcing mesh. Beyond this, the pieces should get larger. This information aids in predicting the size and hence the range of the missiles formed by the breakup and throwout of such targets. Beside the uncertainties about the laws of pressure and of energy absorption, this analysis entirely neglects the flexural wave set up in the concrete. It is entirely possible that there may exist at a certain distance a species of traveling wave amplification between the incident seismic earth wave and the flexural wave it excites in the concrete highway. It is likely, however, that this will be obscured by other effects.

Phenomena in the wall and foundation targets will be much more complex, and no estimates have been made of the shattering to be expected.

2.3 VENTING, AIR BLAST, AND CLOUD RISE

Venting is assumed to occur when the shock reaches the surface, is reflected, and meets the expanding gas bubble. From the theory it is predicted that this should occur about 2-1/2 feet below the surface, and about 0.2 milliseconds should elapse between the shock reaching the surface and the emergence of the gases. The gas velocity is difficult to estimate with much assurance, but it appears that the Laval nozzle relation gives 12,400 fps. Another estimate is based on the expansion of the earth gas bubble in which the pressure at the boundary is utilized, up to the instant of venting, in accelerating the earth beyond. The bubble boundary velocity is approximately doubled on venting to about 30,000 fps. A third estimate is based on calculations of the pressure at venting; with approximately 200,000 psi at venting,

the corresponding velocity is about 7400 fps. A fourth estimate is based on energy considerations. By integration, using Equation 1, the work done against the earth up to venting is about 50 per cent of the total energy release. Then from Bethe's relation that the kinetic energy is $(\gamma - 1)/\gamma$ of the bubble energy, taken as half the original, the jet velocity turns out to be 60,000 fps. While these four values are not consistent, it appears safe to assume that the initial jet velocity will be well over 10,000 fps.

The temperature at venting may be calculated from the equation of state of earth gas, assuming no mass added by further engulfment, and using average densities (isothermal sphere). A value of 10,000° K is obtained. At Bikini Baker (twice the scaled charge depth of the underground nuclear test) little fire was observed. Although the underground nuclear shot is shallower, the density and greater heat of vaporization of earth conspire to absorb more energy than water, so that the appearance of great amounts of fire seems unlikely.

Venting is accompanied by considerable air shock and throwout. The air shock arises from two causes. First, the earth shock reflected at the surface causes the earth to rise, thus acting as an enormous earth piston. The upward velocity of this piston will not decrease too rapidly, since it is being driven from beneath by the gases. Thus the first air shock should show a slow decay from the peak. However, the gases escape with velocity which is considerably greater than that of earth rise, but which decays much faster, since the gases are more easily decelerated. Thus the gas-jet induced air blast should be a high peak followed by a fairly rapid decay.

The maximum time separation of these two shocks should be that between the earth shock reaching the surface and the gases venting. As noted earlier, this is about 0.2 milliseconds. However, the stronger shock will quickly catch up to and coalesce with the weaker one. Hence at positions that are safe for instruments, a single sharp rise should be observed unless the instrument system has response considerably in excess of 5000 cps. This reasoning is based on the experimentally observed behavior of the air blast at Dugway and at the JANGLE HE tests.

In the estimates of jet velocity, it was noted that at venting about half the energy had been used in work against the earth. Thus about 50 per cent can appear in air blast (and cloud rise). From the HE tests, it appears that the effect of air blast was that of 75 per cent of the charge detonated on the surface. Thus the energy in air blast from the underground test will probably be about two-thirds of that from 1 KT of TNT.

By integrating the blast energy flux for the 50 per cent charge in free air, it is found that an equivalent energy of 1.5 per cent of the buried charge does not appear as air blast. It is assumed that this is the energy for cloud rise. The height of rise of the atomic cloud may be estimated from Taylor's relation for a suddenly released source of heat. Using the 1.5 per cent energy, and US Standard Atmosphere meteorological factors appropriate to a 10,000-foot average cloud altitude, we get a rise of about 5000 feet. It is suspected that this is too low, and that two or three times this value may be attained. Of course, the actual meteorological factors rather than those of the standard atmosphere should be used in a comparison with the test. The effect may not be large, since the height of rise depends on the 1/4-power of the factors.

2.4 THROWOUT AND MISSILES

Concomitant with venting are throwout and missile phenomena. In general, results from HE shots lead to the assumption that the matter is ejected along radii from the charge, with a velocity law of the form

$$v = \frac{v_{oo}}{\lambda_c^{\,n}} (\sin \theta)^n , \qquad (12)$$

where v_{oo}/λ_c^n is the zenith velocity, λ_c is the scaled charge depth, θ is the angle between horizontal and a radius from the charge, and n is an exponent probably close to two for JANGLE soil. Theoretically, it is the exponent of pressure decay near the charge, but indirect evidence from observed ranges of missiles at Dugway indicates the exponent is somewhat smaller. The constant v_{oo} is probably between 100 and 400 fps for JANGLE soil, leading to expected zenith velocities from 5000 to 20,000 fps. A full discussion of the implications for throwout and missiles is contained in a forthcoming report; preliminary results have already been communicated, and will not be discussed here.[2]

Some rough estimates of the amount of throwout have been made, based on the material contained in the inverted cone of base equal to the crater opening, and with vertex at the charge. About 30,000 tons of earth should thus be projected radially from the charge. Based on a laboratory analysis, about 2000 tons of this will be below one micron in size, if all the earth has been decohered to its constituent particles. Little grinding action is expected, and the fines resulting should be no more numerous than in the earth and should be considerably less. Thus the 2000 tons is an upper limit to the amount of material

that should remain airborne for a long time after the explosion.

It is also of interest that the energy required to raise 30,000 tons of earth to an average height of, say, 2500 feet is about five per cent of the total release. Hence work against gravity is a small part of the total.

2.5 RETURN

As the throwout returns to the earth, it may be classified as fall-out (ballistic laws); settle-out (Stokes' law); and drift-out (Brownian motion suspension). In the fall-out we include missiles, which are treated in another report.[2]

The settle-out comprises principally those particles in the range of 1 to 100 microns size. Their concentration depends on how thoroughly the explosion and venting processes decohere the earth; no experimental evidence for an estimate is available. Analysis of data from pie-plate collectors can give some order of magnitude information, but this has not been undertaken here. It is fairly certain that most of the fall-out mass is in sizable chunks from one-fourth inch and upwards.

The base surge phenomenon is expected to be present, and should differ but little from predictions based on HE extrapolation. The source of the surge, the column, should show a diameter scaling as $W^{1/3}$ but, owing to the increased effects of gravity and air resistance, the height should be less than that obtained by scaling.

The column is conjectured to be chimney-like in structure, with a hollow core carrying the highly radioactive material ejected from the earth gas bubble. As it collapses to form the surge, the contaminated inner surface is expected to mix turbulently as the surge mushrooms out. However, most of the inner surface should appear at the lower surface of the surge. Surge constants are best estimated from previous nuclear and HE data.

Standard meteorological diffusion theory may be used for predicting the fate of the drift-out, and experience from HE tests should guide the calculations.

2.6 ENERGY PARTITION

From a consideration of the foregoing the probable ultimate energy partition is as follows, in terms of the total release:

Radioactivity	15 %
Pressure wave and plastic flow	30 % or less
Throwout against gravity	5 %
Air blast	48 % or more
Cloud rise	2 %

APPENDIX

LIST OF SYMBOLS

A Coefficient in pressure-radius relation.

A_s Total area of tensile steel in concrete slab.

d Depth of tensile steel in concrete slab.

E Young's modulus of earth.

g Acceleration due to gravity.

h Depth to center of detonation of charge.

n Exponent of radius in pressure-radius relation. Also, exponent in throwout velocity relation.

p Peak or shock pressure.

p_b Pressure in earth-gas bubble at breakaway.

p_i Internal pressure in equation of state of earth.

p_o Reference pressure.

p_t Dynamic tensile strength of earth.

r Horizontal radius from ground zero.

r_a Horizontal limit radius for refracted seismic energy.

r_b Radius of earth gas bubble at breakaway.

r_s Horizontal limit radius for rupture of steel reinforcement in concrete.

t Thickness of concrete slab.

t_b Time to reach breakaway after detonation.

T Temperature.

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T_b	Temperature in earth gas bubble at breakaway.
u	Material velocity.
u_b	Material velocity at breakaway.
U	Shock velocity.
U_b	Shock velocity at breakaway.
v	Velocity of throwout along radius from charge.
v_{oo}	Vertical velocity of throwout for a charge at a scaled depth of unity.
v_1	Seismic velocity of upper stratum of earth.
v_2	Seismic velocity of lower stratum of earth.
w	Width of concrete slab.
W	Weight of TNT (pounds) of given energy release.
W_b	Weight of earth gas bubble at breakaway.
Y	Dynamic tensile strength of earth.
z	Vertical depth coordinate of crater.
z_a	Depth of stratum with higher seismic velocity.
α	Ratio of distance from neutral zone to thickness, in concrete slab.
λ	Scaled horizontal radial distance $r/W^{1/3}$.
λ_c	Scaled depth of charge $h/W^{1/3}$.
ρ	Density at shock in earth gas bubble or in earth.
ρ_o	Reference density.
γ	Exponent for density in equations of state.
σ	Plastic strength of tensile steel in concrete.

ϵ Strain on surface of concrete slab at rupture of tensile steel.

θ Angle between horizontal and radius from charge.

ν Poisson's ratio for earth.

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